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Invariant sets and solutions to higher-dimensional reaction–diffusion equations with source term

Zhu Chunrong [a,b], Qu Changzheng [a,c,*]

[a] Center for Nonlinear Studies, Northwest University, Xi'an 710069, PR China
[b] Department of Mathematics, Anhui Normal University, Wuhu 241000, PR China
[c] Department of Mathematics, Northwest University, Xi'an 710069, PR China

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Abstract

The invariant set and solutions of the two-dimensional reaction–diffusion equation with source term $u_t = A(u)u_{xx} + B(u)u_{yy} + C(u)u_x^2 + D(u)u_y^2 + Q(u)$, is discussed. It is shown that there exist a class of solutions to the equation which belong to the invariant set $E_0 = \{u\colon\ u_x = v_x F(u),\ u_y = v_y F(u)\}$, where v is some smooth function of x and y and F is smooth function of u to be determined. The approach is also developed to deal with the N-dimensional reaction–diffusion equations with source term.

Keywords: Reaction–diffusion equation; Exact solution; Invariant set; Rotation group; Scaling group

1. Introduction

Similarity solutions play important role in characterizing blow up and long time behavior of solutions to nonlinear parabolic equations. There have been a number of interesting results on this work (see [1–3] and references therein). Similarity solutions arise from the scaling invariance of the equations. In [4,5], Galaktionov proposed a "nonlinear" extension to the ordinary scaling group, which is described by the invariance of the set $S_0 = \{u\colon\ u_x = (1/x)F(u)\}$. The extension has been used to construct exact solutions to equations of the form

$$u_t = E\bigl(x, u, u_x, u_{xx}, \ldots, u^{(k)}\bigr), \tag{1}$$

where $u^{(k)}$ denotes the kth-order derivative of u with respect to x. This approach is also related to the sign-invariant and invariant-subspace methods [6–8]. Qu and Estevez [9] further extended the scaling group to more general form which is governed by the invariant set

$$S_1 = \left\{u\colon\ u_x = \frac{1}{x}F(u) + \epsilon F(u)\left[\exp(n-1)\int^u \frac{1}{F(z)}dz\right]\right\}.$$

This approach has been used successfully to construct solutions to a number of nonlinear evolution equations [9,10].

In this Letter, we extend the Galaktionov's approach to study the two-dimensional nonlinear reaction–diffusion equations with source term by introducing the invariant set $E_0 = \{u\colon\ u_x = v_x F(u),\ u_y = v_y F(u)\}$. The invariant set is a natural generalization to S_0 for one-dimensional case, and it will be used to obtain solutions of two-dimensional reaction–diffusion equations with source term

* Corresponding author.
E-mail address: czqu@nwu.edu.cn (C. Qu).

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in Section 2. In Section 3, we shall discuss the invariant set and solutions of $N(N \geqslant 3)$-dimensional reaction–diffusion equations with source term. Section 4 is a concluding remarks on this work.

2. Two-dimensional reaction–diffusion equations

Consider the two-dimensional reaction–diffusion equations with source term

$$u_t = A(u)u_{xx} + B(u)u_{yy} + C(u)u_x^2 + D(u)u_y^2 + Q(u), \tag{2}$$

which has a wide range of physical applications in heat conductivity, combustion and plasma physics, etc. We introduce the invariant set

$$E_0 = \{u\colon\ u_x = v_x F(u), u_y = v_y F(u)\}, \tag{3}$$

where v is some smooth function of x and y, and F is a function to be determined from the invariant condition

$$u(x, y, 0) \in E_0 \Rightarrow u(x, y, t) \in E_0 \quad \text{for } t \in (0, 1].$$

For $u \in E_0$, we obtain solutions of the equation given by

$$\int^u \frac{dz}{F(z)} = v(x, y) + h(t). \tag{4}$$

In the set E_0, we have the following formulas:

$$u_{xx} = v_{xx}F + v_x^2 F'F, \qquad u_{yy} = v_{yy}F + v_y^2 F'F. \tag{5}$$

Suppose Eq. (2) is invariant with respect to the set E_0, and substituting (5) into (2), we obtain

$$h'(t) = Av_{xx} + Bv_{yy} + \frac{Q}{F} + (AF' + CF)v_x^2 + (BF' + DF)v_y^2. \tag{6}$$

Since the left-hand side of (6) does not depend on x and y, differentiating (6) with respect to x and y respectively yields

$$\begin{aligned}
&Av_{xxx} + Bv_{yyx} + \left[A'v_{xx} + B'v_{yy} + \left(\frac{Q}{F}\right)'\right]Fv_x + (AF' + CF)'Fv_x^3 + 2(AF' + CF)v_x v_{xx} + (BF' + DF)'Fv_x v_y^2 \\
&\quad + 2(BF' + DF)v_y v_{yx} = 0, \\
&Av_{xxy} + Bv_{yyy} + \left[A'v_{xx} + B'v_{yy} + \left(\frac{Q}{F}\right)'\right]Fv_y + (AF' + CF)'Fv_x^2 v_y + 2(AF' + CF)v_x v_{xy} + (BF' + DF)'Fv_y^3 \\
&\quad + 2(BF' + DF)v_y v_{yy} = 0.
\end{aligned} \tag{7}$$

This system is different from that for the one-dimensional case [4]. It seems very difficult to determine the coefficient functions in (2) from (7). But we can obtain some results for special $v(x, y)$. Here we consider several special cases:

Case 1. $v_{xy} = 0$.

From $v_{xy} = 0$, we deduce

$$v(x, y) = f(x) + g(y). \tag{8}$$

In this case, E_0 becomes the set $E_0 = \{u\colon\ u_x = f'(x)F(u),\ u_y = g'(y)F(u)\}$. Thus the system (7) reads

$$\begin{aligned}
&Af''' + \left[A'f'' + B'g'' + \left(\frac{Q}{F}\right)'\right]Ff' + (AF' + CF)'Ff'^3 + 2(AF' + CF)f'f'' + (BF' + DF)'Ff'g'^2 = 0, \\
&Bg''' + \left[A'f'' + B'g'' + \left(\frac{Q}{F}\right)'\right]Fg' + (AF' + CF)'Ff'^2 g' + (BF' + DF)'Fg'^3 + 2(BF' + DF)g'g'' = 0.
\end{aligned} \tag{9}$$

Subcase 1.1. Scaling group

Setting $f(x) = \ln|x|$, $g(y) = \ln|y|$, i.e. $f'(x) = 1/x$, $g'(y) = 1/y$. E_0 then becomes the set $S_0 = \{u\colon\ u_x = (1/x)F(u),\ u_y = (1/y)F(u)\}$. This is a extension to the scaling group for the one-dimensional case in [4]. Substituting the expressions for f and g into (9), we obtain

$$\left[2(A - AF' - CF) - (A - AF' - CF)'F\right]\frac{1}{x^3} - (B - BF' - DF)'F\frac{1}{xy^2} + \left(\frac{Q}{F}\right)'F\frac{1}{x} = 0,$$