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An extension of the theorems of three-tangents and three-normals

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Abstract

Fujiwara et al. [T. Fujiwara, H. Fukuda, A. Kameyama, H. Ozaki, M. Yamada, J. Phys. A 37 (2004) 10571] obtained simple geometrical theorems on the three-body problem with zero-angular momentum. These are the so-called three-tangents and three-normals theorems. We extend these theorems to the planar three-body problem of non-zero angular momentum. We have found that the area of the triangle formed with three tangents (respectively, three normals) is expressed in a simple form using the area of the triangle formed with the three momentum vectors, and the angular momentum (respectively, the time derivative of the moment of inertia) of the system.

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Keywords: Three-body problem; Angular momentum; Time derivative of the moment of inertia

1. Introduction

Recently the figure-eight solution to the planar equal-mass three-body problem and related choreographic solutions to the N-body problem attract much attention [1–4]. The figure-eight solution has zero angular momentum. Fujiwara and coworkers [5–7] found various simple geometrical properties of the motion of triple systems with zero angular momentum, but not with equal masses, and above all, they obtained three-tangents theorem and three-normals theorem. Here, a tangent is a line containing the momentum vector, and a normal is a line containing the vector normal to the momentum vector.

Theorem 1 *(Three tangents). If both the linear momentum and the angular momentum are zero, three tangents at the bodies meet at a point or three tangents are parallel.*

Theorem 2 *(Three normals). If the linear momentum is zero and the moment of inertia is constant, three normals at the bodies meet at a point or three normals are parallel.*

We here extend these theorems to the case of non-zero angular momentum. We consider the planar problem. Let us introduce notation. We take the inertial coordinate system and let the origin be the center of mass of the three bodies m_1, m_2, and m_3. The coordinates of the bodies are denoted by $\mathbf{q}_1$, $\mathbf{q}_2$, and $\mathbf{q}_3$. Conjugate momenta are denoted by $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$.

The integral of motion of the center of mass gives us

$$\sum_i \mathbf{p}_i = 0. \tag{1}$$

Eq. (1) is the only condition necessary for the theorems to be true. In fact, in the proof, there appear no quantities obtained from $\mathbf{q}_i$ by differentiating twice with respect to time. This means that the actual form of equations of motion is not important. As is the case of Theorems 1 and 2, ours are rather of geometrical character. Then we may apply these theorems to triple systems with a wide class of equations of motion. One example may be a triple system in dissipative media with resisting force which keeps the center of mass fixed. We have been trying

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to find physical applications. However, we are not successful to find these until now.

We use the following quantities

$$\mathbf{L} = \sum_i \mathbf{q}_i \wedge \mathbf{p}_i = \sum_i \mathbf{L}_i, \quad L = |\mathbf{L}|, \tag{2}$$

and

$$I = \sum_i m_i \mathbf{q}_i^2, \qquad \dot{I} = \sum_i \mathbf{q}_i \cdot \mathbf{p}_i = \sum_i \dot{I}_i. \tag{3}$$

Let us define the double area, $\boldsymbol{\alpha}$, of the triangle formed with three momentum vectors

$$\boldsymbol{\alpha} = \mathbf{p}_1 \wedge \mathbf{p}_2 = \mathbf{p}_2 \wedge \mathbf{p}_3 = \mathbf{p}_3 \wedge \mathbf{p}_1 \quad \text{and} \quad \alpha = |\boldsymbol{\alpha}|, \tag{4}$$

where $\wedge$ indicates the outer product of two vectors.

We introduce two triangles. The three tangents at the bodies generally meet at three points. We denote by $\mathbf{S}^t$ the vectorial area of the directed triangle formed with tangents and let $S^t = |\mathbf{S}^t|$. The three normals at the bodies generally meet at three points. We denote by $\mathbf{S}^n$ the vectorial area of the directed triangle formed with three normals and let $S^n = |\mathbf{S}^n|$. Then, our results are

Theorem 3 *(Extended three-tangents). Let L be the absolute value of the total angular momentum of a general planar three-body problem under the force satisfying Eq. (1). Then, the area $\mathbf{S}^t$ of the triangle formed with three tangents at the bodies is given by*

$$\mathbf{S}^t \cdot \boldsymbol{\alpha} = \frac{L^2}{2}. \tag{5}$$

Theorem 4 *(Extended three-normals). Let $\dot{I}$ be the rate of change of the moment of inertia of a general planar three-body problem under the force satisfying Eq. (1). Then, the area $\mathbf{S}^n$ of the triangle formed with three normals at the bodies is given by*

$$\mathbf{S}^n \cdot \boldsymbol{\alpha} = \frac{\dot{I}^2}{2}. \tag{6}$$

In Section 2, we prove Theorems 3 and 4. In Section 3, theorems are extended to three dimensions.

2. Proof of the theorems

2.1. Proof of Theorem 3

Let v_i^t, $i = 1, 2, 3$ be the vertices of the triangle formed with three tangents such that v_i^t is at the opposite side of m_i with respect to the line connecting m_j and m_k ($j \neq i$ and $k \neq i$). Hereafter, we use the convention that $\{i, j, k\}$ means the cyclic permutation of $\{1, 2, 3\}$. Let $\mathbf{C}_i^t$ be the vector from the origin to v_i^t. Then, the vectorial area $\mathbf{S}^t$ of this triangle can be written as

$$\begin{aligned}\mathbf{S}^t &= \frac{1}{2}\left|\left(\mathbf{C}_i^t - \mathbf{C}_j^t\right) \wedge \left(\mathbf{C}_j^t - \mathbf{C}_k^t\right)\right| \\ &= \frac{1}{2}\left|\mathbf{C}_i^t \wedge \mathbf{C}_j^t + \mathbf{C}_j^t \wedge \mathbf{C}_k^t + \mathbf{C}_k^t \wedge \mathbf{C}_j^t\right|.\end{aligned} \tag{7}$$

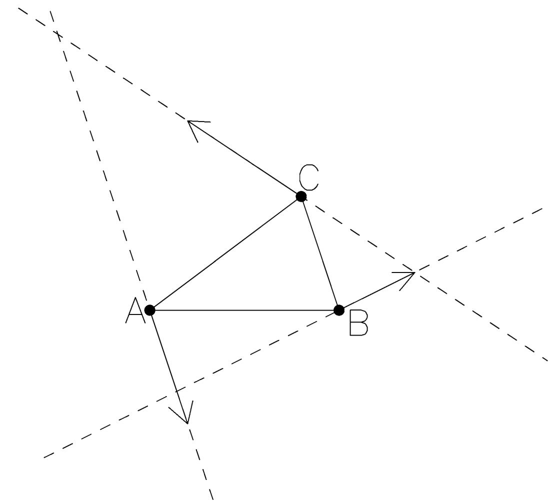

Fig. 1. Geometry of the triangle (solid) in the configuration space and the triangle (dashed) formed with three-tangents. Arrows represent momentum vectors.

By the definition of $\mathbf{C}_i^t$,

$$\left(\mathbf{C}_i^t - \mathbf{q}_j\right) \wedge \mathbf{p}_j = 0 \quad \text{and} \quad \left(\mathbf{C}_i^t - \mathbf{q}_k\right) \wedge \mathbf{p}_k = 0, \quad i = 1, 2, 3, \tag{8}$$

or

$$\mathbf{C}_i^t \wedge \mathbf{p}_j = \mathbf{q}_j \wedge \mathbf{p}_j \equiv \mathbf{L}_j \quad \text{and} \quad \mathbf{C}_i^t \wedge \mathbf{p}_k = \mathbf{q}_k \wedge \mathbf{p}_k \equiv \mathbf{L}_k, \quad i = 1, 2, 3. \tag{9}$$

From this we obtain

$$\begin{aligned}\mathbf{C}_1^t \alpha &= L_3 \mathbf{p}_2 - L_2 \mathbf{p}_3, \\ \mathbf{C}_2^t \alpha &= L_1 \mathbf{p}_3 - L_3 \mathbf{p}_1, \\ \mathbf{C}_3^t \alpha &= L_2 \mathbf{p}_1 - L_1 \mathbf{p}_2.\end{aligned} \tag{10}$$

A straightforward manipulation gives us

$$\mathbf{S}^t \cdot \boldsymbol{\alpha} = \frac{L^2}{2}.$$

2.2. Proof of Theorem 4

Similar to the three-tangents theorem, let v_i^n, $i = 1, 2, 3$ be the vertices of the triangle formed with three normals such that v_i^n is at the opposite side of m_i with respect to the line connecting m_j and m_k. Let $\mathbf{C}_i^n$ be the vector from the origin to v_i^n. Then, the vectorial area $\mathbf{S}^n$ of this triangle can be written as

$$\begin{aligned}\mathbf{S}^n &= \frac{1}{2}\left|\left(\mathbf{C}_i^n - \mathbf{C}_j^n\right) \wedge \left(\mathbf{C}_j^n - \mathbf{C}_k^n\right)\right| \\ &= \frac{1}{2}\left|\mathbf{C}_i^n \wedge \mathbf{C}_j^n + \mathbf{C}_j^n \wedge \mathbf{C}_k^n + \mathbf{C}_k^n \wedge \mathbf{C}_i^n\right|.\end{aligned} \tag{11}$$

By the definition of $\mathbf{C}_i^n$ ($i = 1, 2, 3$),

$$\left(\mathbf{C}_i^n - \mathbf{q}_j\right) \cdot \mathbf{p}_j = 0 \quad \text{and} \quad \left(\mathbf{C}_i^n - \mathbf{q}_k\right) \cdot \mathbf{p}_k = 0, \tag{12}$$

or

$$\mathbf{C}_i^n \cdot \mathbf{p}_j = \mathbf{q}_j \cdot \mathbf{p}_j \equiv \dot{I}_j \quad \text{and} \quad \mathbf{C}_i^n \cdot \mathbf{p}_k = \mathbf{q}_k \cdot \mathbf{p}_k \equiv \dot{I}_k, \quad i = 1, 2, 3. \tag{13}$$