Physics Letters A 372 (2008) 4870–4873

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Semiclassical localization and uncertainty principle

F. Pennini [a,b], A. Plastino [b,*], G.L. Ferri [c], F. Olivares [a]

[a] Departamento de Física, Universidad Católica del Norte, Casilla 1280, Antofagasta, Chile
[b] La Plata Physics Institute, Exact Sciences Fac., National University, Argentina's National Research Council (CCT-CONICET), C.C. 727, (1900) La Plata, Argentina
[c] Facultad de Ciencias Exactas, Universidad Nacional de La Pampa, Peru y Uruguay, Santa Rosa, La Pampa, Argentina

ARTICLE INFO

Article history:
Received 22 February 2008
Received in revised form 5 May 2008
Accepted 7 May 2008
Available online 17 May 2008
Communicated by C.R. Doering

PACS:
2.50.-r
3.67.-a
05.30.-d

Keywords:
Uncertainty principle
Classical localization
Escort distributions
Semiclassical methods

ABSTRACT

Semiclassical localizability problems in phase space constitute a manifestation of the uncertainty principle, one of the cornerstones of our present understanding of Nature. We revisit the subject here within the framework of the celebrated semiclassical Husimi distributions and their associated Wehrl entropy. By recourse to the concept of escort distributions, a well-established statistical concept, we show that it is possible to significantly improve on the current phase-space classical-localization power, thus approaching more closely than before the bounds imposed by Husimi's thermal uncertainty relation.

1. Introduction

The uncertainty principle (UP) poses a rigorous bound to the possibility of semiclassical localization in phase-space. We show here that the current bound of $\hbar$ [1,2] can be improved by recourse to an important statistical tool: escort distributions of a given order [3]. More specifically, we attempt to refine UP-related lower bounds with reference to escort distributions (ED) of order q and *show that semi-classical localization can be significantly improved using them, allowing for a closer approach to the* (*localization-related*) *UP bound* than the one reported in the literature [1,2].

The ED, although a well-established tool, is still (for physicists) a relatively new concept that is rapidly gaining acceptance in the physics world. The semi-classical methodology for applying/relating the concomitant "escort distribution" concepts, joining them with those of information measures (IF) expressed in *phase-space* vocabulary, is detailed, for instance, in Ref. [4]. This usage of IFs will be shown to be of interest below. As one well knows, the oldest and most elaborate phase-space (PS) formulation of quantum mechanics is that of Wigner [5–7]: to every quantum state a PS function (the Wigner one) can be assigned. This phase-space function can assume negative values so that it is considered a quasi-probability density. The negative-values' aspect was circumvented by Husimi [8] (among others), in terms of the so-called Husimi probability distributions $\mu(x,p)$.[1] (Note that whole of quantum mechanics can be completely reformulated in Husimi-terms [9,10].) The distribution $\mu(x,p)$ can be regarded as a "smoothed Wigner distribution" [6]. Indeed, $\mu(x,p)$ is a Wigner-distribution D_W, smeared over an $\hbar$ sized region (cell) of phase-space [2]. The smearing renders $\mu(x,p)$ a positive function, even if D_W does not have such a character. The semi-classical Husimi probability distribution refers to a *special type* of probability: that for simultaneous but approximate location of position and momentum in phase-space [2].

We will in this communication address the issue of q-escort-generalizing Husimi functions in order to show that one can improve, using these generalizations, on this smearing-degree by diminishing the above referred to cell-size $\hbar$. We emphasize the following items: (i) it will be shown that *methods for localization in phase space* can be improved by suitably choosing the escort order q and (ii) for the purpose one needs to bring into the game the purely quantum concept of participation ratio of a mixed state, an interesting instance in which purely quantum concepts already appear

[1] Note that the Husimi distribution function is not, strictly speaking, a probability density because the marginal distribution on each variable is not the squared modulus of a wave function.

* Corresponding author.
E-mail address: plastino@fisica.unlp.edu.ar (A. Plastino).

0375-9601/$ – see front matter
doi:10.1016/j.physleta.2008.05.028

at the semiclassical level. We are of course motivated by the fact that understanding the emergence of classical behavior is one of the major problems of contemporary physics [11]. It should further be emphasized that the subject of phase space localization is of crucial importance in the fascinating field of Quantum Chaos (see, for example, [12–15] and references therein).

2. Escort distributions

Consider two (normalized) probability distributions $f(x)$, $f_q(x)$, and an "operator" $\hat{O}^q$ linking them in the fashion

$$f_q(x) = \hat{O}^q f(x) = \frac{f(x)^q}{\int \mathrm{d}x\, f(x)^q}. \tag{1}$$

We say that $f_q(x)$ is the order q-associated escort distribution of f, with $q \in \Re$. Often, f_q is often able to discern in better fashion than f important details of the phenomenon at hand [3,4].

The expectation value of a quantity $\mathcal{A}$ evaluated with a q-escort distribution will be denoted by $\langle \mathcal{A}\rangle_{f_q}$. For some physical applications of the concept in statistical mechanics see, for instance (not an exhaustive list), [4,16–18], and references therein. For physicists, the fundamental reference on escort distributions is [3].

3. Semi-classical Husimi distributions and Wehrl entropy

Here we wish to introduce the ED-tool into a scenario involving semiclassical Husimi distributions and thereby try to gather *semi-classical information* from escort Husimi distributions (q-HDs).

Some ideas that we abundantly use below can be found in Ref. [18]. In this work we also discuss an important information instrument expressed in *phase-space vocabulary*, namely, the semi-classical Wehrl entropy W, a useful *measure of localization in phase-space* [1,19]. It is built up using coherent states $|z\rangle$ [2,20] and constitutes a powerful tool in statistical physics. Of course, coherent states are eigenstates of a general annihilation operator $\hat{a}$, appropriate for the problem at hand [20–22], i.e.,

$$\hat{a}|z\rangle = z|z\rangle, \tag{2}$$

with z a complex combination of the phase-space coordinates x, p ($\hat{a}$ is nor Hermitian),

$$z = z(x, p) = Ax + iBp, \tag{3}$$

with A, B being $\hat{a}$-depending constants. The pertinent W-definition reads

$$W = -\int \mathrm{d}\Omega\, \mu(x, p) \ln \mu(x, p), \quad \mathrm{d}\Omega = \mathrm{d}x\, \mathrm{d}p/2\pi\hbar, \tag{4}$$

clearly a Shannon-like measure [23] to which MaxEnt considerations can be applied. W is expressed in terms of distribution functions $\mu(x, p)$, *the leitmotif of the present work*, commonly referred to as Husimi distributions [8]. As an important measure of localization in phase-space, W possesses a lower bound, related to the uncertainty principle and demonstrated by Lieb with reference to the harmonic oscillator coherent states [24]

$$W \geqslant 1, \tag{5}$$

on which we wish here to improve upon.

Husimi's μ's are the diagonal elements of the density operator $\hat{\rho}$, that yields all the available physical information concerning the system at hand [25], in the coherent state basis $\{|z\rangle\}$ [20], i.e.,

$$\mu(x, p) \equiv \mu(z) = \langle z|\hat{\rho}|z\rangle. \tag{6}$$

Thus, they are "semi-classical" phase-space distribution functions associated to the system's $\hat{\rho}$ [20–22]. The distribution $\mu(x, p)$ is normalized in the fashion

$$\int \mathrm{d}\Omega\, \mu(x, p) = 1. \tag{7}$$

It is shown in Ref. [26] that, in the all-important harmonic oscillator instance, the associated Husimi distribution reads

$$\mu(x, p) \equiv \mu(z) = \left(1 - e^{-\beta\hbar\omega}\right) e^{-(1-e^{-\beta\hbar\omega})|z|^2}, \tag{8}$$

with $\beta = 1/k_B T$, T the temperature, which leads to a pure Gaussian form in the $T = 0$-limit.

Of course, it is clear that the q-escort Husimi distribution $\gamma_q(x, p)$ will read

$$\gamma_q(x, p) = \hat{O}^q \mu(x, p) = \frac{\mu(x, p)^q}{\int \mathrm{d}\Omega\, \mu(x, p)^q}. \tag{9}$$

4. Participation ratio

What q-values (order of the escort distribution) make physical sense? Surprisingly enough, in order to answer this question recourse has to be made to a purely quantal concept [4]. The quantum physics concept of "degree of purity" of a general density operator (and then of a mixed state) is expressed via $\mathrm{Tr}\,\hat{\rho}^2$ [27]. Its inverse, the so-called participation ratio is

$$\mathcal{R}(\rho) = \frac{1}{\mathrm{Tr}\,\hat{\rho}^2}, \tag{10}$$

an indicator of the number of pure state-projectors that enter $\hat{\rho}$. This expression is particularly convenient for calculations [28]. For pure states $\hat{\rho}^2 = \hat{\rho}$ and $\mathcal{R}(\hat{\rho}) = 1$. If we have mixed states, it is always true that

$$\mathcal{R}(\hat{\rho}) \geqslant 1 \tag{11}$$

because $\hat{\rho}^2 \neq \hat{\rho}$ and $\mathrm{Tr}\,\hat{\rho}^2 \leqslant 1$. Remembering the z-notation introduced in (6), we evaluate Eq. (10) (a "semi-classical" participation ratio) in the following form

$$\mathcal{R}(\mu) = \frac{1}{\int \mathrm{d}\Omega\, \mu(z)^2}, \tag{12}$$

where $\mu(z) = \langle z|\hat{\rho}|z\rangle$ is, of course, the Husimi function and the phase-space volume $\mathrm{d}\Omega$ in classical mechanics is related to differential element $\mathrm{d}^2 z$ in the form $\mathrm{d}\Omega = \mathrm{d}^2 z/\pi$ [20]. We extend now, of course, the $\mathcal{R}$-definition now to γ_q of (9), i.e.,

$$\gamma_q(z) = \frac{\mu(z)^q}{\int \frac{\mathrm{d}^2 z}{\pi} \mu(z)^q}, \tag{13}$$

with $\mu(z)$ the ordinary Husimi distribution. The associated "semi-classical" participation ratio writes then

$$\mathcal{R}_q(\gamma_q) = \frac{1}{\int \frac{\mathrm{d}^2 z}{\pi} \gamma_q(z)^2}, \tag{14}$$

that we can compute explicitly for the harmonic oscillator (HO) using the result given in [18] for the HO-escort-Husimi distribution μ_{HO}. One gets [18]

$$\gamma_q(z) = q\left(1 - e^{-\beta\hbar\omega}\right)^{1-q} \left(\mu(z)\right)^q. \tag{15}$$

Enforcing Eq. (11), that *necessarily holds* from the definition (14), and taking its logarithm, we obtain the inequality

$$2qG(T) - 2\ln(q) \geqslant 3G(T) - \ln 2, \tag{16}$$

with

$$G(T) = \ln\left(1 - e^{-\beta\hbar\omega}\right). \tag{17}$$

We immediately realize now that Eq. (16) *cannot be satisfied by arbitrary q-values*, which leads to important consequences. Returning to (17) we realize that, for instance, when the temperature goes to zero, $G(0) = 0$, and we see that

$$1 \leqslant q \leqslant \sqrt{2}, \tag{18}$$