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The generalized Yablonskii–Vorob'ev polynomials and their properties

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ABSTRACT

Rational solutions of the generalized second Painlevé hierarchy are classified. Representation of the rational solutions in terms of special polynomials, the generalized Yablonskii–Vorob'ev polynomials, is introduced. Differential–difference relations satisfied by the polynomials are found. Hierarchies of differential equations related to the generalized second Painlevé hierarchy are derived. One of these hierarchies is a sequence of differential equations satisfied by the generalized Yablonskii–Vorob'ev polynomials.

1. Introduction

In this Letter we are interested in rational solutions and special polynomials associated with the generalized second Painlevé hierarchy $P_2^{(N)}[\alpha_N, t_1, \dots, t_N]$ [1–3]:

$$\left(\frac{d}{dz} + 2w\right) \sum_{m=1}^{N} t_m L_m\left[w_z - w^2\right] - zw - \alpha_N = 0,$$
$$t_N \neq 0, \quad N > 0. \tag{1.1}$$

Here α_N, $t_1, \dots, t_N$ are parameters and the sequence of operators $L_m[u]$ satisfies the Lenard recursion relation [4]

$$d_z L_{m+1}[u] = \left(d_z^3 + 4u d_z + 2u_z\right) L_m[u], \quad L_0[u] = \frac{1}{2}. \tag{1.2}$$

The first member of the hierarchy is the second Painlevé equation (P_2) up to a simple change of variables. The hierarchy appears as similarity reduction of the modified Korteweg–de Vries (mKdV) hierarchy and possesses several mathematical and physical applications [1,2].

The aim of this Letter is to derive special polynomials $V_n^{(N)}(z; t_1, \dots, t_N)$ related to rational solutions of the hierarchy (1.1). Originally these polynomials were introduced by Yablonskii and Vorob'ev for the rational solutions of P_2 and now are commonly referred to as the Yablonskii–Vorob'ev polynomials [5,6]. In our designations Yablonskii and Vorob'ev dealt with the polynomials $V_n^{(1)}(z; 1)$. Analogous special polynomials for the equations in the hierarchy with $t_1 = \dots = t_{N-1} = 0$, $t_N = 1$ were introduced in [7] and studied in [8,9]. It turns out that the special polynomials $V_n^{(N)}(z; t_1, \dots, t_N)$ possess a lot of properties similar to that of polynomials associated with rational and algebraic solutions of the third, the fourth, and the fifth Painlevé equations [10–15]. First of all they arise as the so-called tau functions of the associated rational solutions. Further they satisfy a certain number of recurrence relations. One of these relations is the following

$$V_{n+1} V_{n-1} = V_n^2 \left(z - 2 \sum_{m=1}^{N} t_m L_m \left[2 \frac{d^2}{dz^2} \ln V_n\right]\right),$$
$$V_n \equiv V_n^{(N)}(z, t_1, \dots, t_N). \tag{1.3}$$

For the equations $P_2^{(1)}[\alpha_1, 1]$, $P_2^{(2)}[\alpha_2, 0, 1]$ it was found in works [6], [16] accordingly. While for the hierarchy $P_2^{(N)}[\alpha_N, 0, \dots, 1]$—in [9]. As far as all other equations in the generalized second Painlevé hierarchy are concerned this relation seems to be new. Note that in the case $N = 1$ a simple substitution transforms this recurrence relation to the Toda lattice equation. Thus the sequence of relations (1.3) with $N > 1$ can be regarded as analogues of the latter.

Further we introduce the so-called generalized P_{34} hierarchy, related to the generalized second Painlevé hierarchy by a Bäcklund transformation and give a representation of its rational solutions via the generalized Yablonskii–Vorob'ev polynomials. In addition

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we obtain an hierarchy of differential equations satisfied by the polynomials.

This Letter is organized as follows. In Section 2 we recall some properties of the generalized P_2 hierarchy and introduce the special polynomials associated with its rational solutions. In Section 3 we obtain differential–difference relations satisfied by the polynomials and study the structure of their roots. Finally in Section 4 we derive some hierarchies related to the generalized P_2 hierarchy including the hierarchies possessing solutions expressible in terms of the generalized Yablonskii–Vorob'ev polynomials.

2. Rational solutions of the generalized second Painlevé hierarchy

Let us briefly review some facts concerning the equations in the generalized P_2 hierarchy. Suppose $w(z) \equiv w^{(N)}(z;\alpha_N)$ is a solution of the equation $P_2^{(N)}[\alpha_N, t_1, \dots, t_N]$. Note that when it does not cause any contradiction we omit the dependence of $w^{(N)}(z;\alpha_N)$ on $t_1, \dots, t_N$. The following transformations

$$S: \quad w^{(N)}(z; -\alpha_N) = -w, \tag{2.1}$$

$$T_\pm: \quad w^{(N)}(z; \alpha_N \pm 1) = -w + \frac{2\alpha_N \pm 1}{2R_N[\mp w_z - w^2] - z}, \quad \alpha_N \neq \mp 1/2, \tag{2.2}$$

produce solutions of the corresponding equation [2,3,17]. In (2.2) $R_N[u]$ denotes the operator

$$R_N[u] \overset{\text{def}}{=} \sum_{m=1}^{N} t_m L_m[u]. \tag{2.3}$$

These transformations in the case $N = 1$ and $t_1 = 1$ were first found in [18]. The Bäcklund transformations (2.1), (2.2) allow us to construct sequences of rational solutions for the hierarchy since each equation possesses the trivial solution $w^{(N)}(z;0) = 0$. Apart from the transformations under consideration we need another relation for the solutions of the equation $P_2^{(N)}[\alpha_N, t_1, \dots, t_N]$:

$$\frac{d}{dz} w^{(N)}(z;\alpha_N + 1) - \left(w^{(N)}(z;\alpha_N+1)\right)^2 = -\frac{d}{dz} w^{(N)}(z;\alpha_N) - \left(w^{(N)}(z;\alpha_N)\right)^2. \tag{2.4}$$

This relation can be verified by a direct substitution of the Bäcklund transformations (2.1), (2.2) [19]. The rational solutions of the generalized P_2 hierarchy are classified in the following theorem.

Theorem 2.1. *The equation $P_2^{(N)}[\alpha_N, t_1, \dots, t_N]$ possesses rational solutions if and only if $\alpha_N = n \in \mathbb{N}$. For fixed values of the parameters $t_1, \dots, t_N$ each rational solution is unique and has the form*

$$w^{(N)}(z;n) = \frac{d}{dz} \ln \frac{V_{n-1}^{(N)}(z)}{V_n^{(N)}(z)}, \quad n > 0,$$
$$w^{(N)}(z;-n) = -w^{(N)}(z;n), \tag{2.5}$$

where $V_n^{(N)}(z) \equiv V_n^{(N)}(z; t_1, \dots, t_N)$ are polynomials. The only remaining rational solution is the trivial solution $w^{(N)}(z;0) = 0$.

Proof. Let $w^{(N)}(z;\alpha_N)$ be a rational solution of the Nth equation in the generalized P_2 hierarchy. The asymptotic analysis of the function $w^{(N)}(z;\alpha_N)$ shows that it has only simple poles with integer residues in the interval $[-N, N]$. By n_l we denote the amount of poles with residue l $(-N \leqslant l \leqslant N,\ l \neq 0)$. The Laurent expansion of the supposed rational solution in a neighborhood of its pole z_0 is the following

$$w^{(N)}(z;\alpha_N) = \frac{l}{z - z_0} + o(1), \quad z \longrightarrow z_0. \tag{2.6}$$

The point $z = \infty$ is a holomorphic point of the rational solution $w^{(N)}(z;\alpha_N)$. The expansion of $w^{(N)}(z;\alpha_N)$ around infinity has the form

$$w^{(N)}(z;\alpha_N) = -\frac{\alpha_N}{z} + o\left(\frac{1}{|z|^2}\right), \quad z \longrightarrow \infty. \tag{2.7}$$

Using the formula for the total sum of the residues of a meromorphic function in the complex plane we get the correlation

$$\sum_{l=-N,\, l\neq 0}^{N} n_l l = -\alpha_N. \tag{2.8}$$

The left-hand side in (2.8) is integer. Hence the necessary condition for the equation $P_2^{(N)}[\alpha_N, t_1, \dots, t_N]$ to possess rational solutions is $\alpha_N \in \mathbb{Z}$. The sufficiency of this condition can be established applying the Bäcklund transformations (2.1), (2.2) to the trivial solution $w^{(N)}(z;0) = 0$. The uniqueness of the rational solution $w^{(N)}(z;\alpha_N)$ at fixed values of the parameters $t_1, \dots, t_N$ follows from the uniqueness of the trivial solution. The representation via polynomials can be obtained by induction with the help of the Bäcklund transformations (2.1), (2.2) and the asymptotic analysis of the solution $w^{(N)}(z;n)$. For the case $N = 1$ see [5,6,20,21]. Similar theorems for other Painlevé-type differential equations were proved in [16,22]. □

The first non-trivial solution of the equations in the hierarchy is $w^{(N)}(z;1) = -z^{-1}$. Thus it can be set $V_0^{(N)}(z; t_1, \dots, t_N) = 1$, $V_1^{(N)}(z; t_1, \dots, t_N) = z$. Without loss of generality we consider all the polynomials to be monic. Making use of the technique suggested in [8] it can be shown that the degree of the nth generalized Yablonskii–Vorob'ev polynomial does not depend on N and is equal to $n(n+1)/2$. We would like to note that sometimes it is convenient to use the polynomials containing the whole sequence of the parameters $\{t_k\}_{k=1}^{\infty}$: $V_n(z;\mathbf{t}) \equiv V_n(z; t_1, \dots, t_N, t_{N+1}, \dots)$. In view of this $V_n^{(N)}(z; t_1, \dots, t_N) = V_n(z; t_1, \dots, t_N, 0, \dots)$.

Several explicit examples of the generalized Yablonskii–Vorob'ev polynomials will be given later.

3. Recurrence relations for the generalized Yablonskii–Vorob'ev polynomials

Yablonskii and Vorob'ev while studying the rational solutions of the second Painlevé equation found several relations for the polynomials used to represent the rational solutions [5,6]. Further several new relations were added to those obtained by Yablonskii and Vorob'ev [13,23,24]. Nowadays these relations being rediscovered several times are as a rule written with the help of the Hirota operator defined by

$$D_z^m f(z) \cdot g(z) = \left[\left(\frac{d}{dz_1} - \frac{d}{dz_2}\right)^m f(z_1) g(z_2)\right]_{z_1 = z_2 = z}, \quad m \in \mathbb{N} \cup \{0\}. \tag{3.1}$$

In this section we show that a part of the relations in question are also valid for the generalized Yablonskii–Vorob'ev polynomials. Furthermore we derive a new one which can be regarded as a generalization of the relation previously known for the polynomials suggested by Yablonskii and Vorob'ev.

Theorem 3.1. *The generalized Yablonskii–Vorob'ev polynomials $V_n^{(N)}(z; t_1, \dots, t_N)$ satisfy the following differential–difference relations*

$$D_z^2 V_{n+1} \cdot V_n = 0, \tag{3.2}$$

$$D_z V_{n+1} \cdot V_{n-1} = (2n+1) V_n^2, \tag{3.3}$$

where D_z is the Hirota operator (3.1).