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Perturbation treatment of symmetry breaking within random matrix theory ☆

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ABSTRACT

We discuss the applicability, within the random matrix theory, of perturbative treatment of symmetry breaking to the experimental data on the flip symmetry breaking in quartz crystal. We found that the values of the parameter that measures this breaking are different for the spacing distribution as compared to those for the spectral rigidity. We consider both two-fold and three-fold symmetries. The latter was found to account better for the spectral rigidity than the former. Both cases, however, underestimate the experimental spectral rigidity at large L. This discrepancy can be resolved if an appropriate number of eigenfrequencies is considered to be missing in the sample. Our findings are relevant for symmetry violation studies in general.

The study of wave chaos using acoustic resonators [1,2] supplies an invaluable additional test of Random Matrix Theory (RMT) [3,4]. In a 1996 paper Ellegaard et al. [5], studied the gradual breaking of the presumed twofold flip symmetry of a quartz crystal by removing an octant of a sphere of an increasing radius at one of the corners and analysing the statistics of the resulting acoustic eigenfrequencies. They found a gradual evolution of the spacing distribution from that of two uncoupled Gaussian Othogonal Ensembles (2GOE) when the crystal is an uncut perfect rectangle, into a single GOE, when a large chunk of the crystal is removed from one of the corners of the rectangle. This constituted a complete breaking of the symmetry present in the crystal in the uncut situation. The spectral rigidity, measured by Dyson's $\Delta_3(L)$ was also measured in this reference. The 2 uncoupled GOEs were found to underestimate by a great amount the large-L data. This was attributed to pseudointegrable trajectories that do not suffer from the symmetry breaking. This point was further analysed by [6]. Using techniques developed by Pandey [7], Leitner [8] treated the symmetry breaking problem with RMT-perturbation. He addressed only the spacing distribution. This work was further extended to the spectral rigidity in [9]. In all of the above treatment of the data of [5], the assumption was made that the uncut crystal has a twofold flip symmetry and thus is describable by two uncoupled GOEs. The treatment of Leitner [8] is found to describe fairly well the NNL distribution, but fails for the spectral rigidity, in contrast to the exact numerical simulation using the Deformed Gaussian Orthogonal Ensemble [10], recently performed in [11]. In this Letter we further analyse the perturbative treatment of symmetry breaking within RMT. We find that the data of [5] can be accounted for with 3GOEs which are gradually mixed till a 1GOE limit is attained. We further find that if some levels were missing in the sample of eigenfrequencies whose statistics is analysed, the $\Delta_3(L)$ can be very well accounted for even at large L without the need for pseudointegrable trajectories, whose calculation is difficult.

Using appropriate perturbative methods Leitner [8] was able to find a formula for the nearest neighbor distribution (NND) which contains the symmetry braking term. He started basically with the formula for the nearest neighbour spacing distribution for the superposition of m GOE's block matrices [3]

$$P_m(s) = \frac{d^2}{ds^2} E_m(s) \tag{1}$$

where, for the case of all block matrices having the same dimension one has

$$E_m(s) = \left(E_1\left(\frac{s}{m}\right)\right)^m, \tag{2}$$

$$E_1(x) = \int_x^\infty \big(1 - F(t)\big)\, dt, \tag{3}$$

$$F(t) = \int_0^t P_1(z)\, dz. \tag{4}$$

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In the above $P_1(z)$ is the normalized nearest neighbour spacing distribution of one block matrix. It is easy to find for $P_m(s)$, the following

$$P_m(s) = \frac{1}{m}\left[\left(E_1\left(\frac{s}{m}\right)\right)^{m-1} P_1\left(\frac{s}{m}\right) + (m-1)\left(E_1\left(\frac{s}{m}\right)\right)^{m-2}\left(1 - F\left(\frac{s}{m}\right)\right)^2\right] \tag{5}$$

$$\equiv P_m^{(1)}(s) + P_m^{(2)}(s). \tag{6}$$

If all the block matrices belong to the GOE, then one can use the Wigner form for $P_1(z)$

$$P_1(z) = \frac{\pi}{2} z e^{-\frac{\pi}{4}z^2} \approx \frac{\pi}{2} z, \tag{7}$$

thus

$$F_1(z) = 1 - e^{-\frac{\pi}{4}z^2} \approx \frac{\pi}{4} z^2, \tag{8}$$

$$E_1(z) = \operatorname{erfc}\left(\frac{\sqrt{\pi}}{2} z\right) \approx 1 - z, \tag{9}$$

where the large-z limits of Eqs. (7)–(9) are also indicated above. It is now clear that the above expression for $P_m(s)$, (5) and (6), contains a term $P_m^{(1)}(s)$ with level repulsion, indicating short-range correlation among levels pertaining to the same block matrix and a second term $P_m^{(2)}(s)$ with no level repulsion, implying short-range correlation among NND levels pertaining to different blocks. Notice that for very small spacing, $P_m(s)$ behaves as

$$P_m(s) \approx \frac{\pi}{2m^2} s + \frac{m-1}{m} \tag{10}$$

for $m = 1$, we get the usual $P_1(0) = 0$, while for $m > 1$, we get $P_m(0) = (m-1)/m$.

To account for symmetry breaking, Leitner [8] considered the mixing between levels pertaining to nearest neighbour block matrices and entails using the 2×2 $P(s)$ distribution with full mixing. The DGOE result for the 2×2 matrix was derived in [12] and the resulting $P(s)$ is a product of a Poissonian term times a mixing term. Leitner's procedure [8] amounts to multiply the factor $P_m^{(2)}(s)$ of Eq. (6) by only the mixing term of the 2×2 $P(s)$ of [12] with the mixing parameter Λ given by [7], $\Lambda = \lambda^2 \rho^2$, with λ^2 being the ratios of the variances of the matrix elements within a block matrix to that of matrix elements pertaining to neighbouring off diagonal block matrices, and ρ is the density of eigenfrequencies. Thus, he found, assuming that $\Lambda \ll 1$,

$$P_m(s, \Lambda) = P_m^{(1)}(s) + P_{2\times 2}(s, \Lambda) P_m^{(2)}(s), \tag{11}$$

where $P_{2\times 2}(s, \Lambda)$ is given by [8]

$$P_{2\times 2}(s, \Lambda) = \sqrt{\frac{\pi}{8\Lambda}}\, I_0\left(\frac{s^2}{16\Lambda}\right) \exp\left(-\frac{s^2}{16\Lambda}\right), \tag{12}$$

where I_0 is the modified Bessel function of order 0. Though $P_m(s)$ is normalized, $P_m(s, \Lambda)$ is not. Accordingly one supplies coefficients c_N and c_D such that

$$P_m(s, \Lambda, c_N, c_D) \equiv c_N P_m(c_D s, \Lambda) \tag{13}$$

is normalized to unity. Similarly, $\langle s \rangle$ should be unity too. Eq. (11) can certainly be generalized to consider the effect of mixing of levels pertaining to next to nearest neighbour blocks, and accordingly, $P_{3\times 3}(s, \Lambda)$, given in Ref. [12] would be used in Eq. (11) instead of $P_{2\times 2}(s, \Lambda)$. In the following, however, we use Eqs. (11), (13) as Leitner did [8].

In [8], Leitner also obtained approximate expression for the spectral rigidity $\Delta_3(L)$ using results derived by French et al. [13]. Leitner's approximation to Δ_3 is equal to the GOE spectral rigidity plus perturbative terms, that is

$$\Delta_3^{(m)}(L; \Lambda) \approx \Delta_3(L; \infty) + \frac{m-1}{\pi^2}\left[\left(\frac{1}{2} - \frac{2}{\epsilon^2 L^2} - \frac{1}{2\epsilon^4 L^4}\right) \times \ln\left(1 + \epsilon^2 L^2\right) + \frac{4}{\epsilon L} \tan^{-1}(\epsilon L) + \frac{1}{2\epsilon^2 L^2} - \frac{9}{4}\right], \tag{14}$$

where

$$\epsilon = \frac{\pi}{2(\tau + \pi^2 \Lambda)}. \tag{15}$$

For the cut off parameter we use the value [9] $\tau = c_m e^{\pi/8 - \gamma - 1}$, where $c_m = m^{m/(m-1)}$ and $\gamma \approx 0.5772$ is Euler's constant. This choice guarantees that when the symmetry is not broken, $\Lambda = 0$, $\Delta_3^{(m)}(L, 0) = m\Delta_3(L/m, \infty)$. In Ref. [14], Leitner fitted Eq. (13) for $m = 2$ to the NND from Ref. [5], however, he did not fit the spectral rigidity. It is often the case that there are some missing levels in the statistical sample analysed. Such a situation was addressed recently by Bohigas and Pato [15]. These authors have started from the general expression of $\Delta_3(L)$ derived by Dyson and Mehta [4], namely,

$$\Delta_3(L) = \frac{L}{15} - \frac{1}{15 L^4} \int_0^L dx\, (L - x)^3 \left(2L^2 - 9xL - 3x^4\right) Y_2(x), \tag{16}$$

where the two-point cluster function, $Y_2(x_1, x_2)$, which owing to translational invariance becomes a function of the difference $x = |x_1 - x_2|$, is defined by the usual expression,

$$Y_2(x_1, x_2) = 1 - \frac{R_2(x_1, x_2)}{R_1(x_1) R_1(x_2)}, \tag{17}$$

where R_2 is the 2-point correlation function and R_1 is the density of the spectrum.

If a fraction, $1 - g$, of the levels were actually analysed, the cluster function remains invariant, apart from a rescaling of the relevant variables, when the unfolded spectrum is employed, namely

$$Y_2^g(x_1, x_2) = 1 - \frac{(1-g)^2 R_2(x_1^g, x_2^g)}{(1-g) R_1(x_1^g)(1-g) R_1(x_2^g)} = Y_2\left(x_1^g, x_2^g\right), \tag{18}$$

where the scaled variables x_i^g are just $\frac{x_i}{(1-g)}$.

Using the above equation for the cluster function in the general expression for $\Delta_3(L)$, we obtain the Missing-Level (ML) expression of [15]

$$\Delta_3^g(L) = g \frac{L}{15} + (1-g)^2 \Delta_3\left(\frac{L}{1-g}\right). \tag{19}$$

In the application to our current problem of m-coupled GOE's, the above formula continue to be valid since the basic input into its derivation, namely the invariance of Y_2, apart from the scaling of the argument x into x^g, is quite general. Accordingly, we have the desired ML formula of $\Delta_3(L)$ for m-coupled GOEs,

$$\Delta_3^{(m)g}(L; \Lambda) = g \frac{L}{15} + (1-g)^2 \Delta_3^{(m)}\left(\frac{L}{1-g}; \Lambda\right). \tag{20}$$

The presence of the linear term, even if small, could explain the large L behavior of the *measured* $\Delta_3(L)$. We call this effect the Missing-Level (ML) effect. Another possible deviation of Δ_3 from Eq. (14) could arise from the presence of pseudo-integrable effect (PI) [6,16]. This also modifies Δ_3 by adding a Poisson term just like Eq. (19).

The results of our analysis are shown in Figs. 1 and 2. In Fig. 1, the sequence of six measured NNDs were fitted for $m = 2$ and $m = 3$. It can be seen that the Leitner model with three coupled GOEs give a comparable and in some cases even better fit than the