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Synchronization of weighted networks and complex synchronized regions

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Abstract

Since the Laplacian matrices of weighted networks usually have complex eigenvalues, the problem of complex synchronized regions should be investigated carefully. The present Letter addresses this important problem by converting it to a matrix stability problem with respect to a complex parameter, which gives rise to several types of complex synchronized regions, including bounded, unbounded, disconnected, and empty regions. Because of the existence of disconnected synchronized regions, the convexity characteristic of stability for matrix pencils is further discussed. Then, some efficient methods for designing local feedback controllers and inner-linking matrices to enlarge the synchronized regions are developed and analyzed. Finally, a weighted network of smooth Chua's circuits is presented as an example for illustration.

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Keywords: Network synchronization; Complex synchronized region; Matrix pencil; Weighted network

1. Introduction and problem formulation

Synchronization of symmetrically and diffusively coupled networks has been well studied (see [1–5] and references therein). After [6], it has been known that the synchronizability of symmetrically and diffusively coupled networks is completely determined by the synchronized region derived from the master stability function and an eigenratio of the corresponding Laplacian matrix. Within this framework, the synchronization problem of networks with different topological structures has been extensively studied e.g. in [1,4,5]. The relationship between the network synchronizability and network structural parameters has also been studied e.g. in [7–9]. The synchronized region problem was then studied; for example, the ragged synchronized regions were found via simulations in [10], the coexistence of unbounded and bounded synchronized regions was analyzed in [11], and the existence of an arbitrary number of disconnected synchronized regions was studied in [12]. Compared with symmetrical and unweighted networks, weighted networks have not been investigated in detail [13], especially for non-symmetrically weighted networks [14]. Since a weighted Laplacian matrix usually has complex eigenvalues, the problem of complex synchronized regions for weighted networks should be studied carefully, which is the attempt of the present Letter.

Consider a general dynamical network consisting of N coupled identical nodes, with each node being an n-dimensional dynamical system, described by

$$\dot{x}_i = f(x_i) - c\sum_{j=1}^{N} a_{ij} H(x_j), \quad i = 1, 2, \ldots, N, \tag{1}$$

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where $x_i = (x_{i1}, x_{i2}, \dots, x_{in}) \in \mathbb{R}^n$ is the state vector of node i, $f(\cdot): \mathbb{R}^n \to \mathbb{R}^n$ is a smooth vector-valued function, $c \in \mathbb{R}$ is an constant representing the coupling strength, $H(\cdot): \mathbb{R}^n \to \mathbb{R}^n$ is called the inner linking function, and $A = (a_{ij})_{N\times N}$ is called the outer coupling matrix, which represents the coupling configuration of the entire network. Suppose network (1) is diffusively coupled, i.e.,

$$a_{ii} = -\sum_{j=1, j\neq i}^{N} a_{ij}, \quad i = 1, 2, \dots, N.$$

Further, if the graph corresponding to A is connected, i.e., A is irreducible, then zero is an eigenvalue of A with multiplicity 1. In this Letter, $a_{ij} < 0$ is not necessarily equal to a_{ji} or -1; that is, A is a Laplacian matrix corresponding to a weighted graph. Suppose the eigenvalues of A are $\lambda_1 = 0, \lambda_2 = \alpha_2 + \beta_2 i, \dots, \lambda_N = \alpha_N + \beta_N i$. By the graph theory, one knows that all the real parts of the nonzero eigenvalues of A are larger than zero, which are denoted by

$$0 < \alpha_2 \leqslant \alpha_3 \leqslant \cdots \leqslant \alpha_N. \tag{2}$$

In [6], A was supposed to be diagonalizable, but it was pointed out in [14] that the diagonalizability of A is not necessary for network synchronization. In this Letter, the diagonalizability of A is not assumed either. In addition, in all existing references, the coupling strength c is supposed to be larger than zero. From a mathematical viewpoint, c can be less than zero as well, as shown below in this Letter.

The dynamical network (1) is said to achieve (asymptotical) synchronization if

$$x_1(t) \to x_2(t) \to \cdots \to x_N(t) \to s(t), \quad \text{as } t \to \infty,$$

where, because of the diffusive coupling configuration, the synchronous state $s(t) \in \mathbb{R}^n$ is a solution of an individual node, satisfying $\dot{s}(t) = f(s(t))$. Here, $s(t)$ can be an equilibrium point, a periodic orbit, or even a chaotic orbit.

As shown in [6], the stability of the synchronous solution $x_1(t) = x_2(t) = \cdots = x_N(t) = s(t)$ can be determined by analyzing the following equation, known as the master stability equation:

$$\dot{\omega} = \bigl[Df\bigl(s(t)\bigr) + \sigma DH\bigl(s(t)\bigr)\bigr]\omega, \tag{3}$$

where $\sigma \in \mathbb{C}$, and $Df(s(t))$ and $DH(s(t))$ are the Jacobian matrices of functions f and H at $s(t)$, respectively.

The largest Lyapunov exponent $L_{\max}$ of network (1), which can be calculated from system (3) and is a function of σ, is referred to as the master stability function. In addition, the region $S \subseteq \mathbb{C}$ of complex parameter σ where $L_{\max}$ is negative is called the synchronized region of network (1). Based on the results of [6,15], the synchronous solution of network (1) is asymptotically stable if, and only if,

$$-c(\alpha_k + \beta_k i) \in S, \quad k = 2, 3, \dots, N. \tag{4}$$

Note that, in a symmetrical network, where A is symmetrical, the synchronized region S is an interval or the union of several intervals on the real axis. But in a directed weighted network, the eigenvalues of A are generally complex numbers. Only for symmetrical networks, which have been well studied, eigenvalues of A are all real, so that the eigenratio λ_2/λ_N characterizes the network synchronizability: the larger the λ_2/λ_N, the better the synchronizability. But for directed weighted networks, it is more complicated. The real and imaginary parts of the eigenvalues of A need to be considered simultaneously.

If the synchronous state is an equilibrium point, then $Df(s(t))$ and $DH(s(t))$ reduce to real constant matrices, denoted by F and H, respectively. In this case, system (3) becomes

$$\dot{\omega} = [F + \sigma H]\omega. \tag{5}$$

Hence, the synchronized region S becomes the stable region of the matrix pencil $F + \sigma H$ with respect to the complex parameter σ. This Letter only studies this case when the synchronous state is an equilibrium point.

The rest of the Letter is organized as follows. In Section 2, several types of complex stable regions for the matrix pencil $F + \sigma H$ are studied. In Section 3, the convexity characteristic for the stability of the matrix pencil is discussed based on the classical Lyapunov function method. In Section 4, some design methods for feedback controllers and inner-linking matrices to enlarge the synchronized regions are presented. In Section 5, a weighted network of smooth Chua's circuits is simulated to illustrate the theoretical results. The Letter is concluded by the last section.

2. Types of complex stable regions of the matrix pencil

In symmetrical networks, the synchronized region changes over the real axis, which can be bounded, unbounded, empty or a union of several intervals. In this section, the asymmetrical case is investigated.

As mentioned above, when the synchronous state is an equilibrium state, the synchronized region problem reduces to a stability problem of the matrix pencil $F + \sigma H$, where F and H are real matrices, and σ is a complex parameter. It turns out that the latter setting is easier to discuss technically. To proceed, the following lemma [16,17] is needed.