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The improved sub-ODE method for a generalized KdV–mKdV equation with nonlinear terms of any order

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Abstract

In this Letter, Li and Wang's sub-ODE method [X.Z. Li, M.L. Wang, Phys. Lett. A 361 (2007) 115] is improved and applied to the generalized KdV–mKdV equation with nonlinear terms of any order. As a result, more travelling wave solutions are obtained including not only all the known solutions found by Li and Wang but also other formal solutions. This improved sub-ODE method can be used for solving other nonlinear partial differential equations with nonlinear terms of any order in mathematical physics.

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Keywords: Sub-ODE method; Nonlinear partial differential equations; Travelling wave solutions

1. Introduction

It is well known that nonlinear partial differential equations (NLPDEs) are widely used to describe complex phenomena in various fields of science from physics to biology, chemistry, mechanics, etc. As mathematical models of the phenomena, the investigation of exact solutions of NLPDEs will help one to understand these phenomena better. In the past several decades, many effective methods for obtaining exact solutions of NLPDEs have been presented such as the inverse scattering method [1], Hirota's bilinear method [2], Bäcklund transformation [3], Painlevé expansion [4], sine–cosine method [5], homogeneous balance method [6], homotopy perturbation method [7–9], variational iteration method [10–13], asymptotic methods [14,15], non-perturbative methods [16], Adomian decomposition method [17], Jacobi elliptic function expansion method [18–20], Exp-function method [21–35], algebraic method [36–38], F-expansion method [39–45], auxiliary equation method [46–51], tanh-function method [52–60], and so on.

The last four methods mentioned above belong to a class of method called subsidiary ordinary differential equation method (sub-ODE method for short) [36]. The key ideas of the sub-ODE method are that the travelling wave solutions of the complicated NLPDE can be expressed as a polynomial, the variable of which is one of the solutions of simple and solvable ODE that called the sub-ODE, and the degree of the polynomial can be determined by balancing the highest derivatives with nonlinear terms in the considered NLPDE. The sub-ODEs which were often used are the Riccati equation, Jacobi elliptic equation, projective Riccati equations, etc. With the development of computer science, recently, the sub-ODEs with nonlinear terms of high order have attracted much attention [36–51]. This is due to the availability of symbolic computation systems like Mathematica or Maple which enable us to perform the complex and tedious computation on computers.

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In order to solve NLPDEs involving higher order nonlinear terms, recently, Li and Wang [61] proposed a sub-ODE method by introducing a subsidiary ODE including an arbitrary positive power

$$F'(\xi) = AF(\xi) + BF^{2+p}(\xi) + CF^{2+2p}(\xi), \quad p > 0, \tag{1}$$

and its five special solutions, here A, B and C are constants, the prime denotes $\mathrm{d}/\mathrm{d}\xi$.

Li and Wang's method has been applied to some important NLPDEs [61–63], the present Letter is motivated by the desire to improve the work made in [61] by introducing more solutions of Eq. (1) including not only all the solutions given in [61] as special cases but also other formal solutions.

The rest of this Letter is organized as follows. In Section 2, we introduce some special solutions of Eq. (1) to improve Li and Wang's method. In Section 3, we use these special solutions to solve the generalized KdV–mKdV equation with nonlinear terms of any order [61]. In Section 4, some conclusions are given.

2. Special solutions of the subsidiary ODE

In order to find some solutions of Eq. (1) conveniently, we set

$$F(\xi) = G^{\frac{1}{p}}(\xi), \tag{2}$$

then Eq. (1) becomes

$$G'^2(\xi) = Ap^2G^2(\xi) + Bp^2G^3(\xi) + Cp^2G^4(\xi). \tag{3}$$

With the aid of Eqs. (2) and (3), we can easily find some special solutions of Eq. (1), which are listed as follows.

2.1. If $A > 0$, then Eq. (1) has the following hyperbolic function solutions:

$$F_\pm(\xi) = \left\{\frac{\pm 2A\,\mathrm{sech}(p\sqrt{A}(\xi+\xi_0))}{\sqrt{B^2-4AC} \mp B\,\mathrm{sech}(p\sqrt{A}(\xi+\xi_0))}\right\}^{\frac{1}{p}}, \quad B^2-4AC>0, \tag{4}$$

$$F_\pm(\xi) = \left\{\frac{\pm 2A\,\mathrm{csch}(p\sqrt{A}(\xi+\xi_0))}{\sqrt{4AC-B^2} \mp B\,\mathrm{csch}(p\sqrt{A}(\xi+\xi_0))}\right\}^{\frac{1}{p}}, \quad B^2-4AC<0, \tag{5}$$

$$F_\pm(\xi) = \left\{-\frac{A}{B}\left[1 \pm \tanh\left(\frac{p\sqrt{A}}{2}(\xi+\xi_0)\right)\right]\right\}^{\frac{1}{p}}, \quad B^2-4AC=0, \tag{6}$$

$$F_\pm(\xi) = \left\{-\frac{A}{B}\left[1 \pm \coth\left(\frac{p\sqrt{A}}{2}(\xi+\xi_0)\right)\right]\right\}^{\frac{1}{p}}, \quad B^2-4AC=0. \tag{7}$$

2.2. If $A < 0$, then Eq. (1) has the following trigonometric function solutions:

$$F_\pm(\xi) = \left\{\frac{\pm 2A\sec(p\sqrt{-A}(\xi+\xi_0))}{\sqrt{B^2-4AC} \mp B\sec(p\sqrt{-A}(\xi+\xi_0))}\right\}^{\frac{1}{p}}, \quad B^2-4AC>0, \tag{8}$$

$$F_\pm(\xi) = \left\{\frac{\pm 2A\csc(p\sqrt{-A}(\xi+\xi_0))}{\sqrt{B^2-4AC} \mp B\csc(p\sqrt{A}(\xi+\xi_0))}\right\}^{\frac{1}{p}}, \quad B^2-4AC>0. \tag{9}$$

2.3. If $A = 0$, then Eq. (1) has the following rational solutions:

$$F(\xi) = \left\{\frac{4B}{B^2p^2\xi^2-4C}\right\}^{\frac{1}{p}}, \tag{10}$$

$$F_\pm(\xi) = \left\{\pm\frac{1}{p\sqrt{C}\xi}\right\}^{\frac{1}{p}}, \quad B=0, \quad C>0. \tag{11}$$

Setting A, B and C in above solutions as special cases, all the solutions given by Li and Wang in [61] of Eq. (1) can be recovered one by one. If we set $B < 2A$, $C = \frac{B^2}{4A} - A$ and $\xi_0 = 0$, then Eq. (4) becomes

$$F_\pm(\xi) = \left\{\frac{\pm 1}{\cosh(p\sqrt{A}\xi)) \mp \frac{B}{2A}}\right\}^{\frac{1}{p}}, \tag{12}$$

where $F_+(\xi)$ is the solution (2.2) given by Li and Wang in [61].