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Physics Letters A 351 (2006) 246–250
PHYSICS LETTERS A
www.elsevier.com/locate/pla

Matrix oscillator and Laughlin Hall states

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Received 19 September 2005; accepted 2 November 2005
Available online 9 November 2005
Communicated by V.M. Agranovich

Abstract

We propose a quantum matrix oscillator as a model that provides the construction of the quantum Hall states in a direct way. A connection of this model to the regularized matrix model introduced by Polychronakos is established. By transferring the consideration to the Bargmann representation with the help of a particular similarity transformation, we show that the quantum matrix oscillator describes the quantum mechanics of electrons in the lowest Landau level with the ground state described by the Laughlin-type wave function. The equivalence with the Calogero model in one dimension is emphasized. It is shown that the quantum matrix oscillator and the finite matrix Chern–Simons model have the same spectrum on the singlet state sector.

PACS: 03.65.Fd; 03.65.Sq; 05.30.Pr

Keywords: Matrix oscillator; Bargmann representation; Laughlin states

1. Introduction

The finding of quantum levels of nonrelativistic electrons in a uniform magnetic field is a well-known problem in quantum mechanics and extends to studying the physics of the quantum Hall effect. The physics of electrons in the lowest Landau level exhibits some interesting features, the example of which is the occurrence of the incompressible fluid like [1] states of condensed electrons whose excitations have fractional charge and obey fractional statistics [2,3]. These states appear only when the electron densities are certain rational fractions of the density corresponding to a fully filled lowest Landau level and the gap in their excitation spectrum gives rise to the experimentally observed fractional quantum Hall effect. They are described by the Laughlin wave functions [4]. The tools for studying the exactness and universality of the Laughlin wave functions are offered in a natural way in the realm of matrix models [5].

One can argue about using noncommutative physics for describing real physical systems, such as the quantum Hall fluid. The natural realization of noncommutative space is provided by the planar coordinates of quantum particles moving in a constant magnetic field. Recently, an attempt was made by Susskind [6] to describe the incompressible quantum Hall fluid in terms of the noncommutative Chern–Simons theory on the plane, the approach that has the connection to an analogy between the physics of electrons in a strong magnetic field and the properties of D-branes in string theory [7]. The dynamics of quantum Hall fluids in the framework of noncommutative field theory was treated in [8,9].

As the Chern–Simons theory on the plane necessarily describes a spatially infinite quantum Hall system, it was also of interest to find a description of finite systems with a finite number of electrons and this was achieved by the model introduced by Polychronakos [10]. Such a regularized model, proposed as a theory of finite matrices with additional boundary vector fields, has provided a description of the quantum Hall droplet and its boundary excitations [11]. The quasiparticle and quasihole states were explained in terms of Schur functions within an algebraic approach [12].

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0375-9601/$ – see front matter
doi:10.1016/j.physleta.2005.11.008

The finite matrix Chern–Simons model is described by two matrices X_1, X_2 or $A, A^\dagger$. It was shown [12] that both these matrices could not be diagonal simultaneously with some operators on the diagonal. This would lead to inconsistencies and to only two towers of states of the Bose and Fermi type, respectively. There was also a problem with the construction of the general Laughlin states [13]. However, the strong connection of the matrix Chern–Simons model with the Calogero model and the quantum Hall effect was pointed out in [10–13].

Recently, a quantum matrix oscillator was proposed and its equivalence to the Calogero-type models was established [14,15]. The classical version of the matrix oscillator was introduced in [16] and the path integral quantization of this model was performed in [17].

In this Letter we propose a quantum matrix oscillator and establish its connection to the finite matrix Chern–Simons model introduced by Polychronakos. We use the matrix oscillator model [14] to find the physical states of electrons in the lowest Landau level. The ground states are Laughlin-type states and the analysis leading to this result, together with the construction of the excited states, relies heavily on the consideration that is carried out in the Bargmann representation. The main point here is to reduce the eigenvalue problem to a much simpler one and then to transfer the obtained results back to the original problem, with the help of a conveniently constructed similarity transformation. Although the analysis is performed for the one-dimensional case only, it can as well be straightforwardly extended to two and higher dimensions as long as identical particles are considered. As a consequence, the results obtained can be analytically continued onto the whole complex plane incorporating in such a way the wave functions of the true Laughlin form that depend on complex variables. The relevance of the matrix oscillator model to the quantum Hall physics has been emphasized throughout the procedure.

The Letter is organized as follows. In the Section 2 we introduce the matrix oscillator model and make a connection to the finite matrix model. The next step is made in Section 3 where the equation of motion stemming from the matrix model action is recognized as the quantization condition imposed on the matrix coordinates of the electrons. After finding the representation of the matrices X_1 and X_2, that solve the quantization condition, in Section 4 we construct the matrix operators required for building up the Fock space of states for the matrix oscillator model. The main result and the crucial analysis of the Letter is contained in Section 5, where the transition to a particularly convenient Bargmann representation is made. This enables us to identify the eigenstates of the matrix oscillator model as the wave functions of physical states describing electrons in the lowest Landau level, including the ground state Laughlin wave function and excitations over the Laughlin state.

2. Matrix oscillator and action

Let us construct an action for the matrix oscillator described by $N \times N$ matrices $X, \mathcal{P}$ with operator-valued matrix elements, $(X_{ij})^\dagger = X_{ji}$, $(\mathcal{P}_{ij})^\dagger = \mathcal{P}_{ji}$; $i, j = 1, 2, \dots, N$. We take the matrix X to be diagonal, with real elements. The Hamiltonian and commutation relations [14] are then ($\hbar = 1$)

$$H = R\left(\frac{1}{2m}\mathcal{P}^2 + \frac{1}{2}m\omega^2 X^2\right)C, \tag{1}$$

$$[X, \mathcal{P}] = \imath\mathcal{V}, \quad \mathcal{V} = (1-\nu)\mathbf{1} + \nu\mathcal{J}, \tag{2}$$

where $R = (1\dots1)$ is a row-vector whose all components are units, and $C = R^T$ is a transpose of R. Also, we have $RC = N$ and $CR = \mathcal{J}$, where $\mathcal{J}$ is the $N \times N$ matrix with units at all positions. The matrix $\mathcal{V}$ is symmetric, $\mathcal{V}^T = \mathcal{V}$, where $\nu > -1/N$ is a real parameter and m is the mass. Generally, $\mathcal{V}$ is a Hermitian matrix $\mathcal{V}^\dagger = \mathcal{V}$, with $\nu_{ii} = 1$ and $\nu_{ij}^* = \nu_{ji}$, $\forall i, j$, and the effective Hamiltonian contains three-body interactions [15].

In order to describe two-dimensional systems of N charged particles with charge e in a magnetic field B, it is convenient to define the matrix $X_1 \equiv X$ and a second matrix X_2 expressed in terms of $\mathcal{P}$ as

$$X_2 = -\frac{1}{eB}\mathcal{P} = -\frac{1}{m\omega}\mathcal{P}, \tag{3}$$

where $\omega = eB/m$. Note that the trace $\mathrm{Tr}[X_1, X_2]$ is equal to $\frac{N}{\imath eB}$, in accordance with the relation (2).

The coordinates of the electrons can be globally parametrized in a fuzzy way by introducing two $N \times N$ Hermitian matrices X_a; $a = 1, 2$. The action leading to the quantum matrix oscillator is then given by the regularized finite matrix Chern–Simons model introduced by Polychronakos

$$S_M = \frac{eB}{2}\int dt\, \mathrm{Tr}\big[\varepsilon_{ab}X_a\big(\dot{X}_b - \imath[A_0, X_b]\big) + 2\theta A_0\big] - \frac{\omega eBN}{2\bar\psi\psi}\int dt\, \bar\psi X_a X_a \psi - \int dt\, \bar\psi(\imath\partial_t + A_0)\psi, \tag{4}$$

where $eB\theta = k$, A_0 is a matrix entering into the above action only linearly and ψ ($\bar\psi = \psi^{*T}$) is a boundary vector field. The action (4) is invariant under the transformations $X_a \to UX_aU^{-1}$, $\psi \to U\psi$, $\bar\psi \to \bar\psi U^{-1}$, $A_0 \to UA_0U^{-1} + \imath U\partial_t U^{-1}$, where U is a unitary matrix, $U \in U(N)$. The term with ω serves as a potential box that keeps particles near the origin and also provides a Hamiltonian for the theory that chooses a unique ground state, while the last term in the action can be interpreted as a boundary term. Also, note that the minor change is made in the harmonic term in respect to the action of Ref. [10], namely $\mathrm{Tr}(X_a)^2$ is replaced by $\bar\psi(X_a)^2\psi$. But, as these two parts yield the same spectrum when acting on the singlet sector of the $U(N)$ group, this replacement essentially does not make any difference. The only reason for replacing the $\mathrm{Tr}(X_a)^2$ by $\bar\psi(X_a)^2\psi$ is that the later gives rise to the quantum Calogero model (in the quantum Calogero model the inverse square potential term has $\nu(\nu+1)$ as a prefactor, with ν being the coupling constant), while the former is related to the classical Calogero model (this has the factor ν^2 in front of the inverse square potential term). Later, we shall see that, after a diagonal form of one of the matrices X_1 or $X_1 + \imath X_2$ is assumed, the boundary fields transform into the R, C matrices, i.e., row and column matrices defined after Eq. (2).