Available online at www.sciencedirect.com
SCIENCE DIRECT®
Physics Letters A 351 (2006) 274–282
PHYSICS LETTERS A
www.elsevier.com/locate/pla

An analytical method to solve heat conduction in layered spheres with time-dependent boundary conditions

Xiaoshu Lu [a,b,*], Martti Viljanen [a]

[a] *Laboratory of Structural Engineering and Building Physics, Department of Civil and Environmental Engineering, Helsinki University of Technology, PL 2100, FIN-02015 Hut, Espoo, Finland*

[b] *Department of Physiology, Finnish Institute of Occupational Health, Topeliuksenkatu 41 a A, FIN-00250 Helsinki, Finland*

Received 30 June 2005; received in revised form 28 October 2005; accepted 7 November 2005

Available online 17 November 2005

Communicated by A.P. Fordy

Abstract

An analytical method is proposed to solve the equation of heat conduction in a layered sphere subject to a time-dependent boundary temperature. It is well known that for such problems in general, eigenvalue and residue computation poses a challenge, which can become too complicated to handle with many layers. In this Letter, the proposed analytical method is free of eigenvalue and residue calculations. A closed-form approximate solution is derived with high accuracy.

PACS: 44.10.+i; 44.05.+e

Keywords: Layered sphere; Heat conduction; Analytical method; Laplace transform

1. Introduction

The study of transient heat conduction in a composite slab is important in physics and engineering areas such as thermodynamics, fuel cells, electrochemical reactors, high density microelectronics, food products, solidification processes and many others. In terms of the geometry of the slab, the layered sphere is extensively used in studying the thermal properties of composite media by assuming that spherical particles are embedded in the composite matrix. Thermal analysis of such a composite matrix has wide applications in manufacturing also. The existing studies are often based on the analytical analysis of heat conduction in a layered sphere. The purpose of this article is to establish a closed-form approximate analytical solution for transient heat conduction in a layered sphere subject to a time-dependent boundary condition.

Analytical methods for the transient heat conduction problem in the spherical geometry are completely analogous to those in Cartesian coordinates. The commonly applied techniques are finite integral transforms which are often employed to a single layer slab, Green's functions, orthogonal expansions and Laplace's transform [1].

A traditional application of the first two techniques often leads to eigenvalue problems. In a single layer slab, for instance, the eigenfunction links the space and time variables when applying separation of variables. However, in a multilayer slab, the eigenfunctions can present boundary conditions at the contacted layers. Hence, eigenvalue problems may exist even for the steady-state heat conduction problem in a composite slab (e.g. [2]). The associated eigenvalue calculations may become much more complicated.

* Corresponding author.
E-mail address: xiaoshu@cc.hut.fi (X. Lu).

0375-9601/$ – see front matter
doi:10.1016/j.physleta.2005.11.017

Similarly, an application of the third technique, the Laplace transform, often yields a residue computation. In a composite slab, the residue computation is found by directly and numerically searching for the roots of a hyperbolic equation, finding the derivatives of the equation, and evaluating and summing the residues. The calculation procedure is tedious if the slab has more than two layers, as numerical searching for roots has to be made with a very fine increment for the inverse Laplace transform to prevent missing roots which can lead to a wrong inverse [3].

Due to the complex nature of the associated eigenvalue and residue problems, the advantage of analytical methods over numerical methods is sometimes hard to recognise. Closed-form solutions are lacking for heat conduction problems in composite slabs though there has been extensive research on this topic. Most of the analytical solutions in literature are available formally: numerical programs for searching eigenvalues and residues are necessary. In Cartesian coordinates, a closed-form solution was given for the transient heat conduction problem in a three-layer composite slab subject to an unchanged temperature [4]. A closed-form solution in an n-layer composite slab was conjectured [4]. In cylindrical coordinates, Fredman developed a semi-analytical method for composite layer diffusion [5]. The method relied on a local analytical solution for a single material layer combined with a numerical solution scheme for the material boundary states [5]. In a spherical geometry, an analytical–numerical technique was employed to predict the temperature and heat flux histories in a bi-layer composite sphere [6].

In a multi-dimensional composite slab, similar analytical techniques are often adopted for the heat conduction problem (e.g. [7] with Green's function). The associated eigenvalue problems are much more complicated as the eigenvalues may become imaginary. Attention must be paid when computing eigenvalues since spacing between successive eigenvalues changes between zero and the maximal value. Numerically, imaginary eigenvalues can produce instability [7]. For the steady-state heat conduction problem, Haji-Sheikh et al. presented an analytical solution in a three-dimensional two-layer slab [8]. In both papers, an effort has been made to develop an efficient algorithm to compute the eigenvalues.

Most of the above-cited papers produce an exact solution for the heat conduction problem. There is no doubt that the exact solutions are extremely valuable whenever they appear. However, all these works need numerical schemes for eigenvalues which are tedious if the slab has more than three layers. Recently, a novel analytical method was developed to solve the one-dimensional transient heat problem in a composite slab with a time-dependent boundary temperature [9]. Unlike most of the traditional methods, the new method involves no numerical computation such as a numerical search for eigenvalues and residues. The boundary conditions are given more generally. An approximate closed-form solution was provided with high accuracy. In this Letter, the developed analytical method is extended to a spherical structure. The objective of this study is to derive a more general closed-form solution for the transient heat conduction in a layered sphere subject to a time-dependent temperature change. It is believed that the results can be beneficial to the study of thermal properties of composite media.

2. Mathematical formulation

2.1. Governing equations

Consider an n-layer sphere having constant thermal conductivity, diffusivity and radius in each layer which are denoted by λ_j, k_j and r_j, $j = 1, \dots, n$. A schematic figure is shown in Fig. 1. The spherical shells are $[r_{j-1}, r_j]$, $j = 1, \dots, n$, with $r_0 = 0$.

The general heat conduction problem for the composite sphere can be described by the following equation for temperatures $T_j(t, r)$ in spherical coordinates:

$$\frac{\partial T_j}{\partial t} = \frac{k_j}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T_j}{\partial r}\right), \quad r \in [r_{j-1}, r_j],\ j = 1, \dots, n. \tag{2.1a}$$

The boundary conditions are given as

$$T_j(t, r) = T_{j+1}(t, r), \quad r = r_j,\ j = 1, \dots, n-1, \tag{2.1b}$$

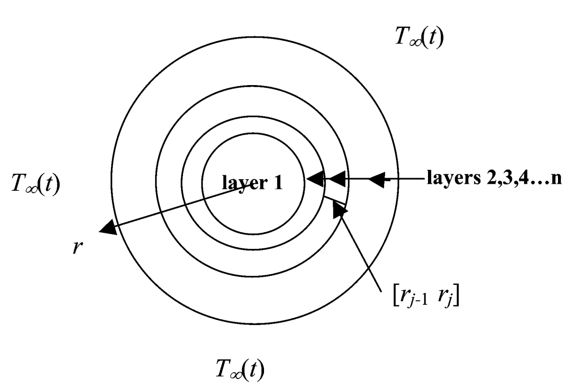

Fig. 1. Schematic diagram of an n-layer composite sphere.