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Physics Letters A 372 (2008) 1236–1239

PHYSICS LETTERS A

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Computing the non-linear anomalous diffusion equation from first principles

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Received 8 June 2007; received in revised form 3 September 2007; accepted 10 September 2007

Available online 14 September 2007

Communicated by A.R. Bishop

Abstract

We investigate asymptotically the occurrence of anomalous diffusion and its associated family of statistical evolution equations. Starting from a non-Markovian process *à la* Langevin we show that the mean probability distribution of the displacement of a particle follows a generalized non-linear Fokker–Planck equation. Thus we show that the anomalous behavior can be linked to a fast fluctuation process with memory from a microscopic dynamics level, and slow fluctuations of the dissipative variable. The general results can be applied to a wide range of physical systems that present a departure from the Brownian regime.

PACS: 05.70.Ln; 05.30.Pr

Considerable interest and effort has recently been applied to the analysis of anomalous behavior in collective motion. Studies in this area go from turbulence [1], granular matters [2] and economic processes [3] to social behavior [4]. An example of the signature of such processes is the well known anomalous diffusion behavior, where the second moment $\langle x(t)^2\rangle \propto t^\alpha$, with $\alpha \neq 1$, is the archetypal quantity of analysis [5]. This communication aims to introduce some insight into the microscopic foundation *à la Langevin* of such ubiquitous behaviors, passing from the equation of motion of a single particle (microscopic dynamic) to the effective statistical properties of the ensemble system (macroscopic laws).

One intriguing aspect in the description of complex systems is the existence of a non-linear Fokker–Planck equation (NLFP)

$$\frac{\partial P}{\partial t} \propto \frac{\partial^2 P^q}{\partial x^2}, \qquad q > 0, \tag{1}$$

which combines aspects that make it very appropriate to treat, at this point phenomenologically, different fields where anomalous behaviors are relevant (like anomalous transport and long tail probability distributions among others). In fact, this NLFP equation has been applied to disordered systems and porous media [6], where the underlying processes present characteristics of self-similarity, scaling laws, etc., as well as to non-extensive statistical mechanics (see [7] and references therein).

Such equation can be derived, in the case of porous media, combining the continuity equation with two *empirical* relations: Darcy's law and a state equation for polytropic gases (or fluids) $p \propto \rho^\nu$; where p is the pressure and ρ the density [8] of these systems. In non-extensive statistical mechanics theory, for the general case, the NLFP equation has been derived employing self consistent approaches [7,9], using Langevin equations which are *themselves* functions of the probability, that is $\dot{x}(t) = \mathcal{F}[x, P(x,t), \eta(t)]$, where $\eta(t)$ is a white noise. Then, taking into account an appropriated stochastic calculus (Ito, Stratonovich, etc.), it is possible to arrive at the NLFP equation written in Eq. (1).

It is worth stressing that this non-linear evolution equation has been extensively applied to the formalism of non-extensive statistical mechanics, where applications in various scientific fields have been reported, including: long range interaction [10], multifractality [11], behavior at the edge of chaos [12], and others (see [7] and references therein). It

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0375-9601/$ – see front matter
doi:10.1016/j.physleta.2007.09.020

is therefore a very important task to give an explanation of such non-linear evolution for the probability distribution, which turns out to be of relevance in many complex systems studies.

The above mentioned methodology for deducing the NLFP equation shows an interesting gap in understanding the fundamental underlying processes that make possible the non-linearity in the evolution equation of this probabilistic function. None of the previous approaches give a clear answer to the problem, apart from showing that those processes can present memory effects at a microscopic level, which may be an important ingredient for the emergence of this particular non-linear (anomalous) evolution.

We will proceed as follows. Starting from one particle dynamics, *à la* Langevin, with memory components, we will be able to infer its asymptotic probability distribution in space and time and the resulting evolution equation. Then, after calculating the average over the slow fluctuations, we will obtain the general expression for the related NLFP equation.

The microscopy dynamics à la Langevin. The presence of a memory kernel in the Langevin equation goes back to works by Kubo, Mori, Nakajima, Zwanzig, et al. (see for example Ref. [13]), but more recently there have been studies showing that a memory kernel is equivalent to the introduction of a fractional differential operator [14]. This has been considered in the Langevin framework, as well as in the Fokker–Planck equation, and makes it possible to describe the anomalous transport process with some degree of accuracy. Let us examine the dynamics of a single particle coupled to a *complex heat bath* with temperature $k_B T$. The equation of motion of such a particle can be written in the form

$$M\ddot{x} + M\int_{0^+}^{t} \gamma(t')\dot{x}(t-t')\,dt' = \xi(t), \tag{2}$$

we have denoted with 0^+ a possible cut-off. Here $\xi(t)$ is a Gaussian long-range correlated noise and $\gamma(t)$ the associated dissipative kernel that can be obtained from the elimination of bath variables [15]. The dissipative kernel $\gamma(t)$ calculated from a microscopic random-matrix model, is

$$Mk_B T\gamma(t) = 2A_0\Gamma(\alpha)\cos\left(\frac{\alpha\pi}{2}\right)t^{-\alpha}, \quad t>0, \tag{3}$$

where the exponent α characterizes, in the non-Ohmic regime, the behavior of the spectral density of the bath at low frequencies [16]. The solution we are looking for is subject to the initial conditions $x(0)=0$ and $\dot{x}(0)=0$. It is worth mentioning here that in Ref. [17] we have introduced a functional approach that enables us to solve this kind of linear memory-like Langevin equation in the presence of any arbitrary noise $\xi(t)$, nevertheless, in the present communication we are interested only in a Gaussian noise.

Note that if we rewrite the equation of motion (2) using a fractional derivative, it is possible to see that it differs from the usual Langevin equation by introducing, in the dissipative term, a fractional differential operator of order $\alpha-1$, for $\alpha\in(0,2)$, with a coefficient $\gamma_\alpha = \pi A_0/[Mk_B T\sin(\alpha\pi/2)]$ being the dissipation parameter due to complex friction model. As we remarked before, $\xi(t)$ is a Gaussian noise with zero mean and correlation $\langle\xi(t)\xi(0)\rangle = 2A_0\Gamma[\alpha]\cos(\alpha\pi/2)t^{-\alpha}$, with $t>0$. A_0 is the coupling strength of the particle with the complex bath. Eq. (2) allows us to obtain several results concerning statistical properties of an ensemble of particles subject to slow fluctuations in their dynamic (dissipative) parameter, as we will describe in detail below. In particular we are interested in the description of the position of the particle at a given time, which can be addressed using the marginal probability distribution $P(x,t)=\int P(x,V,t)\,dV$; where $V(t)\equiv\dot{x}(t)$.

Because the noise is Gaussian, the calculation of the two-dimensional joint probability distribution $P(x,V,t)$ is simply done in terms of a few cumulants, then, following [17], we can calculate the marginal probability distribution $P(x,t)$ knowing the second moment of the position, which is given by

$$\langle x^2(t)\rangle = \frac{2kT}{M}t^2E_{2-\alpha,3}\left(-\gamma_\alpha t^{2-\alpha}\right), \tag{4}$$

where $E_{\mu,\nu}$ is the generalized Mittag–Leffler function [18]. After a transient, the second moment has a clear anomalous behavior, given by [19]

$$\langle x^2(t\to\infty)\rangle \approx \frac{2kT}{M\gamma_\alpha}\frac{t^\alpha}{\Gamma(1+\alpha)} \equiv \frac{t^\alpha}{b}, \tag{5}$$

we have defined $b = M\gamma_\alpha\Gamma(1+\alpha)/(2kT)$. For $\alpha=1$ this result coincides with the well known diffusive behavior obtained from the asymptotic limit of the Ornstein–Uhlenbeck process [5,20]. The solution for the asymptotic marginal probability distribution is given by

$$P(x,t|b) \equiv P(x,t) = \sqrt{\frac{b}{2\pi t^\alpha}}\exp\left(-\frac{bx^2}{2t^\alpha}\right), \tag{6}$$

where we have denoted explicitly the conditional character of the distribution with the parameter b. This quantity, b, can be seen as a *slow-effective dissipative coefficient* for this anomalous process. We mention here that the evolution equation of such an asymptotic processes is given by a diffusion-like equation, as was also pointed out in [21]

$$\frac{\partial P(x,t|b)}{\partial t} = \frac{\alpha t^{\alpha-1}}{2b}\frac{\partial^2 P(x,t|b)}{\partial x^2}. \tag{7}$$

In fact, if we associate $\alpha=2H$ with $\alpha\in(0,2)$, $P(x,t|b)$ is the $1-$ *time* probability distribution of the well known fractional Brownian motion (fBm) process [5,22]. This result shows that the fluctuations at the microscopic level appear as an anomalous dependence in time (anomalous transport), but preserve the Gaussian character of the distribution for fixed times, as expected from the linear Gaussian model (2). For a more general situation see [17].

By introducing a scaling analysis we can calculate the power spectrum from the position of the ensemble of particles. First note that the distribution described by Eq. (6) satisfies the scaling relation

$$P\left(\Lambda^{\alpha/2}x, \Lambda t|b\right) = \Lambda^{-\alpha/2}P(x,t|b), \tag{8}$$