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A two-parameter generalization of Shannon–Khinchin axioms and the uniqueness theorem

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Abstract

Based on the one-parameter generalization of Shannon–Khinchin (SK) axioms presented by one of the authors, and utilizing a tree-graphical representation, we have developed for the first time a two-parameter generalization of the SK axioms in accordance with the two-parameter entropy introduced by Sharma, Taneja, and Mittal. The corresponding unique theorem is also proved. It is found that our two-parameter generalization of Shannon additivity is a natural consequence from the Leibniz product rule of the two-parameter Chakrabarti–Jagannathan difference operator.
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1. Introduction

We often encounter complex systems which obey asymptotic power-law distributions in many fields such as high-energy physics, biophysics, turbulence, scale-free networks, economic science and so on. In order to explain the statistical natures of such complex systems, one of the fundamental approaches is to generalize statistical mechanics in terms of a suitable generalization of the Boltzmann–Gibbs–Shannon (BGS) entropy. Tsallis' non-extensive thermostatistics [1–4] is one of such generalizations. Naudts [5] has developed the generalized thermostatistics based on deformed exponential and logarithmic functions in general context.

In the field of information theory, Sharma and Taneja [6], and independently Mittal [7] obtained a two-parameter entropy in 1975 by generalizing Chaundy and McLeod's functional equation, which characterizes Shannon's entropy. In the field of statistical physics, Borges and Roditi [17] has proposed the two-parameter generalized entropy $S_{\alpha,\beta}$ of Eq. (10) in 1998. Later Kaniadakis, Lissia and Scarfone [8,9] have considered a differential–functional equation imposed by the MaxEnt principle, and obtained the two-parameter (κ and r) entropy,

$$S_{\kappa,r} = -\sum_i p_i^{1+r}\left(\frac{p_i^{\kappa} - p_i^{-\kappa}}{2\kappa}\right), \tag{1}$$

which is found to be equivalent to the Sharma–Taneja–Mittal entropy. For the sake of simplicity Boltzmann's constant k_{B} is set to unity in this Letter. The two-parameter entropy $S_{\kappa,r}$ includes some one-parameter generalized entropies as different special cases, e.g., Tsallis [10], Abe [11] and Kaniadakis [12,13] entropies. For example, when $r = -\kappa$ and $q = 1 - 2\kappa$, $S_{\kappa,r}$ reduces to Tsallis' entropy

$$S_q = \frac{1 - \sum_i p_i^q}{q - 1}, \tag{2}$$

and when $r = 0$, $S_{\kappa,r}$ reduces to Kaniadakis' entropy

$$S_\kappa = \sum_i \frac{p_i^{1-\kappa} - p_i^{1+\kappa}}{2\kappa}. \tag{3}$$

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Consequently the generalization of thermostatistics based on the two-parameter entropy provides us a unified framework of non-extensive thermostatistics. It has been shown that the two-parameter entropy has some important thermostatistical properties, such as positivity, continuity, expansibility, concavity, Lesche stability, and so on [8,9]. Thermodynamic stability for microcanonical systems described by the two-parameter entropy has been studied in Ref. [14]. Scarfone [15] has further developed the Legendre structure among the generalized thermal quantities in the thermostatistics based on the two-parameter entropy $S_{\kappa,r}$.

Abe [11] provided the procedure which generates an entropy functional from the function

$$g(s) \equiv \sum_i p_i^s, \tag{4}$$

where p_i is a probability of ith event. He observed that the BGS entropy is obtained by acting the standard derivative on $g(s)$ as

$$\left[-\frac{dg(s)}{ds}\right]_{s=1} = -\sum_i p_i \ln p_i = S^{\mathrm{BGS}}, \tag{5}$$

and that Tsallis' entropy is obtained by acting Jackson's q-derivative (or q-difference operator),

$$\mathrm{D}_x^q f(x) \equiv \frac{f(qx) - f(x)}{(q-1)x}, \tag{6}$$

on the function g as

$$\left[-\mathrm{D}_s^q g(s)\right]_{s=1} = \frac{1 - \sum_i p_i^q}{q-1} = S_q. \tag{7}$$

At first glance, this seems to be nothing more than a mathematical exercise because it is a dummy variable s on which the derivative operator acts. Johal [16] however gave this a nice physical interpretation in the context of multifractal as follows. According to the box counting algorithm for a multifractal distribution p_i, the phase space of a system is divided into the boxes with the side length ℓ, and a local scaling for the distribution is assumed to be

$$p_i \propto \ell^{\alpha_i}, \tag{8}$$

where α_i is the crowding index. Then Eq. (7) can be expressed as

$$-\sum_i \alpha_i \mathrm{D}_{\alpha_i}^q p_i = \frac{1 - \sum_i p_i^q}{q-1} = S_q. \tag{9}$$

In addition he emphasized that the importance of a difference operator, like the q-derivative, due to the discreteness of the crowding indices α_i for a multifractal distribution in general.

Following Abe's procedure, Borges and Roditi [17] have obtained the two-parameter generalized entropy,

$$\left[-\mathrm{D}_s^{\alpha,\beta} g(s)\right]_{s=1} = \sum_i \frac{p_i^\alpha - p_i^\beta}{\beta - \alpha} = S_{\alpha,\beta}, \tag{10}$$

by using the Chakrabarti and Jagannathan (CJ) difference operator [18]

$$\mathrm{D}_x^{\alpha,\beta} f(x) \equiv \frac{f(\alpha x) - f(\beta x)}{(\alpha - \beta)x}, \quad \alpha, \beta \in R. \tag{11}$$

The two-parameter CJ difference operator $\mathrm{D}_x^{\alpha,\beta}$ includes Jackson's q-derivative as a special case of $\alpha = q$ and $\beta = 1$. It is instructive to see that the following one-parameter difference operator

$$\mathrm{D}_x^\kappa f(x) \equiv \frac{f((1+\kappa)x) - f((1-\kappa)x)}{2\kappa x}, \tag{12}$$

which is another special case ($\alpha = 1-\kappa$ and $\beta = 1+\kappa$) of CJ difference operator, is in accordance with Kaniadakis' entropy S_κ.

Both two-parameter entropies (1) and (10) are equivalent each other, and they are related by

$$\kappa = \frac{\beta - \alpha}{2} \quad \text{and} \quad 1 + r = \frac{\alpha + \beta}{2}. \tag{13}$$

Note that Eq. (11) is symmetric under the interchange of the two parameters $\alpha \leftrightarrow \beta$. Consequently the two-parameter entropy $S_{\alpha,\beta}$ has the same symmetry.

On the other hand, it is well known that BGS entropy can be uniquely determined by the Shannon–Khinchin (SK) axioms [19,20]. During the rapid progress of Tsallis' thermostatistics, the generalized SK axioms were proposed by dos Santos [21] and by Abe [22]. Later, one of the authors [23] has generalized the SK axioms for one-parameter generalization of BGS entropy, and proved the uniqueness theorem rigorously. To the best of our knowledge, there is no generalization of SK axioms for either Kaniadakis' entropy S_κ or the two-parameter entropy $S_{\alpha,\beta}$. Since S_κ is a special case of $S_{\alpha,\beta}$, it is a natural to generalize the SK axioms for the two-parameter entropy $S_{\alpha,\beta}$. This is the main purpose of this Letter. In the next section we review the one-parameter (q) generalization of SK axioms, among which the key ingredient is the q-generalized Shannon additivity. In order to develop a two-parameter generalization of the Shannon additivity, tree-graphical representation is utilized. In Section 3 we prove the uniqueness theorem associated with the obtained two-parameter SK axioms. Some examples of the two-parameter entropies are presented in Section 4. In Section 5 it is shown that the two-parameter generalized Shannon additivity is symmetric under the interchange of the two parameters. The relation with the Leibniz product rule of difference (or derivative) operator is discussed. Final section is devoted to our conclusions.

2. One-parameter generalization of Shannon additivity

We first briefly review the q-generalized Shannon–Khinchin axioms [23], from which the following one-parameter (q) generalization of BGS entropy is uniquely determined:

$$S_q(p_1, \ldots, p_n) = \frac{1 - \sum_{i=1}^n p_i^q}{\phi(q)}, \tag{14}$$

with $q \in R^+$ and $\phi(q)$ satisfies the following properties (i)–(iv):

(i) $\phi(q)$ is continuous and has the same sign as $q-1$;
(ii) $\lim_{q\to 1} \phi(q) = 0$, and $\phi(q) \neq 0$ for $q \neq 0$;

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