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Envelope breather solution and envelope breather lattice solutions to the NLS equation

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Abstract

In this Letter, dependent and independent variable transformations are introduced to solve the nonlinear Schrödinger (NLS) equation systematically by using the knowledge of elliptic equation and Jacobian elliptic functions. It is shown that different kinds of solutions can be obtained to the NLS equation, including many kinds of envelope breather solutions and envelope breather lattice solutions.

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1. Introduction

Among the soliton bearing nonlinear equations, the modified Korteweg–de Vries (mKdV) equation is of special interest [1,2]. For it possesses rich solutions, such as solitary solutions [1–4], periodic solutions [3–6], breather solution [1,2,7,8], breather lattice solutions [7,8]. A particularly interesting type of solution is the so-called breather kind of solution, usually this kind of solutions is unavailable and such solutions have to be solved numerically [7]. In some cases, however, the analytical expressions in closed form can be found, such as the breather lattice solution for the sine-Gordon equation [9] and for the mKdV equation [7,8].

In Refs. [7–9], Kevrekidis's research group has applied some ansatzes to obtain the breather lattice solutions to the mKdV equation and the sine-Gordon equation. The aim of present Letter is to present the envelope breather solutions and envelope breather lattice solutions of the NLS equation in a systematical way. Based on the introduced transformations, we will show systematical results about these breather-type solutions for the NLS equation by using the knowledge of elliptic equation and Jacobian elliptic functions [3,4,10–12].

The cubic nonlinear Schrödinger (NLS) equation reads [1,2]

$$iu_{t'} + \alpha' u_{x'x'} + \beta' |u|^2 u = 0, \tag{1}$$

where $i = \sqrt{-1}$.

Eq. (1) can be transformed as

$$\alpha' \frac{\partial^2 E}{\partial x'^2} + \left(kc - k^2\alpha'\right)E + \beta' E^3 + i\left(\frac{\partial E}{\partial t'} + 2k\alpha' \frac{\partial E}{\partial x'}\right) = 0, \tag{2}$$

with the transformation

$$u(x', t') = E(x', t')e^{ik(x'-ct')}, \tag{3}$$

where the amplitude $E(x', t')$ is a real function of its arguments. Next, we will try to find the analytical expression of $E(x', t')$.

First of all, the real part and imaginary part of Eq. (2) can be separated into a set of equations, i.e.

$$\frac{\partial E}{\partial t'} + 2k\alpha' \frac{\partial E}{\partial x'} = 0, \tag{4a}$$

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$$\alpha' \frac{\partial^2 E}{\partial x'^2} + \left(kc - k^2\alpha'\right)E + \beta' E^3 = 0. \tag{4b}$$

Substituting Eq. (4a) into Eq. (4b) yields the modified KdV (mKdV) equation

$$\frac{\partial E}{\partial t'} + \alpha E^2 \frac{\partial E}{\partial x'} + \beta \frac{\partial^3 E}{\partial x'^3} = 0, \tag{5}$$

with

$$\alpha \equiv \frac{6\alpha'\beta'}{c - k\alpha'}, \qquad \beta \equiv \frac{2\alpha'^2}{c - k\alpha'}. \tag{6}$$

Set $t = t'$, $x = \beta^{-1/3}x'$ and $v = \pm\sqrt{\frac{\alpha}{6}}\beta^{-1/6}E$, Eq. (5) can be rewritten as

$$v_t + 6v^2 v_x + v_{xxx} = 0, \tag{7}$$

which is called the positive mKdV (pmKdV) equation [7,8].

If we set $t = t'$, $x = \beta^{-1/3}x'$ and $v = \pm\sqrt{-\frac{\alpha}{6}}\beta^{-1/6}E$, Eq. (5) can be rewritten as

$$v_t - 6v^2 v_x + v_{xxx} = 0, \tag{8}$$

which is called the negative mKdV (nmKdV) equation [7,8].

From the above relation between (7), (8) and (5), it is obvious that if one derives the solutions to (7) or (8), then the solutions to (5) can be obtained directly by the rescaled independent variables and dependent variable. Next, we will show the details to derive many kinds of solutions to (5), especially the breather solutions, the solutions describing the interaction of two-soliton and the shelf-shaped solutions, which have not been reported in the literature in a systematical way.

2. Envelope breather lattice solutions and envelope breather solutions: Case for the pmKdV equation

In order to obtain the breather solution and breather lattice solution to Eq. (5) satisfied by the amplitude E, the following transformation

$$v = 2\frac{\partial}{\partial x}\tan^{-1}\phi \tag{9}$$

must be introduced, and then ϕ satisfies

$$\left(1 + \phi^2\right)(\phi_t + \phi_{xxx}) + 6\phi_x\left(\phi_x^2 - \phi\phi_{xx}\right) = 0. \tag{10}$$

Eq. (10) can be solved by introducing the following independent variable and dependent variable transformations

$$\xi = ax + bt + \xi_0, \qquad \eta = cx + dt + \eta_0, \tag{11}$$

and

$$\phi = AU(\xi)V(\eta), \tag{12}$$

where ξ_0 and η_0 are two constants, A is a constant to be determined, U and V satisfy the following elliptic equation

$$U_\xi^2 = s_1U^4 + p_1U^2 + q_1,$$
$$V_\eta^2 = s_2V^4 + p_2V^2 + q_2, \tag{13}$$

where s_1, s_2, p_1, p_2, q_1 and q_2 are determined constants.

Substituting (11) and (12) into (10) yields the following algebraic equations

$$b + p_1a^3 + 3p_2ac^2 = 0, \tag{14a}$$
$$q_1A^2a^2 + s_2c^2 = 0, \tag{14b}$$
$$q_2A^2c^2 + s_1a^2 = 0, \tag{14c}$$
$$d + 3p_1a^2c + p_2c^3 = 0. \tag{14d}$$

For the algebraic equations (14), some cases can be addressed, first of all, we will address some special cases.

Case 1. If $s_1 = s_2 = q_1 = q_2 = 0$ and $p_1 > 0$, $p_2 > 0$, then from Eq. (14), we have

$$U = \gamma_1 e^{\pm\sqrt{p_1}\xi}, \qquad V = \gamma_2 e^{\pm\sqrt{p_2}\eta}, \tag{15}$$

where γ_1 and γ_2 are two constants. The solution (12) can be written as

$$\phi_{2-1} = Ae^{\pm(\sqrt{p_1}\xi \pm \sqrt{p_2}\eta)}, \tag{16}$$

where A is a constant, and the solution (16) is a kind of shelf-shaped solution, whose graphical presentation is shown in Fig. 1, an obvious shelf shape.

Case 2. If only $s_2 = q_1 = 0$, then from Eq. (13), we have

$$U_\xi^2 = s_1U^4 + p_1U^2, \qquad V_\eta^2 = p_2V^2 + q_2. \tag{17}$$

Actually, if we set $U = \frac{1}{W}$, then we have

$$W_\xi^2 = p_1W^2 + s_1. \tag{18}$$

It is obvious that V and W satisfy the same equation, this equation has three subcases to be considered.

Case 2a. If $p_1 > 0$, $s_1 > 0$ or $p_2 > 0$, $q_2 > 0$, the solution to U or V is

$$\frac{1}{U} = \pm\sqrt{\frac{s_1}{p_1}}\sinh(\sqrt{p_1}\xi + c_1),$$

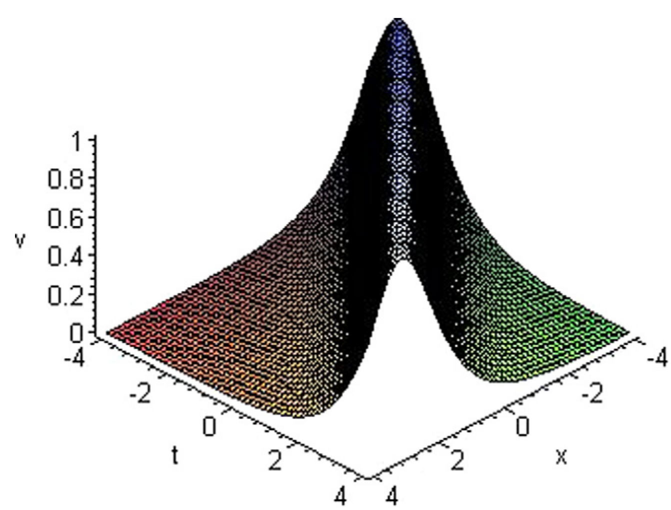

Fig. 1. The graphical presentation shows the space–time evolution of the shelf-shaped solution of Eqs. (15) and (16), where the parameters are chosen as $p_1 = p_2 = A = 1$, $a = b = \xi_0 = 2$, $c = d = \eta_0 = 1$.