Physics Letters A 375 (2011) 3512–3516

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Two-dimensional solvable chaos

Kiyoshi Sogo [a,*], Atsushi Masumizu [b]

[a] Department of Physics, School of Science, Kitasato University, 1-15-1, Kitasato, Minami-ku, Sagamihara, Kanagawa, 252-0373, Japan
[b] Institute of Computational Fluid Dynamics, 1-16-5, Haramachi, Meguro-ku, Tokyo, 152-0011, Japan

ARTICLE INFO

Article history:
Received 3 May 2011
Received in revised form 18 July 2011
Accepted 4 August 2011
Available online 12 August 2011
Communicated by A.P. Fordy

Keywords:
Solvable chaos
Invariant measure
Inverse problem
Random number generator

ABSTRACT

Two methods are proposed to construct two-dimensional chaotic maps. Several examples of exactly solvable chaotic maps and their invariant measures are obtained. They are isomorphic maps of square to square, plane to plane and circle to circle having various symmetry such as uniform, rotational and the quartic rotational symmetry.

1. Introduction

Recently one of the authors gave a series of solutions for *the inverse problem* of one-dimensional chaotic maps [1]. We imply here by the word "inverse problem" the problem to find iterative function $X = \varphi(x)$ showing chaotic behavior which has the prescribed invariant measure $\rho(x)$. Solvable chaotic maps obtained in Ref. [1] are those with invariant measures $\rho(x) \propto 1/\sqrt{1-x^{2n}}$ $(n = 1, 2, 3)$. Among them the simplest case of $n = 1$, which is nothing but the Chebyshev map, is given by

$$X = \cos\bigl(2 \cdot \cos^{-1} x\bigr) = 2x^2 - 1 \quad \bigl(|x| < 1\bigr) \tag{1.1}$$

which satisfies the equation

$$\frac{|\mathrm{d}X|}{\sqrt{1-X^2}} = 2 \cdot \frac{|\mathrm{d}x|}{\sqrt{1-x^2}}. \tag{1.2}$$

This is a typical differential equation of the inverse problem which is used to find a chaotic map $X = \varphi(x)$ having the given invariant measure $\rho(x)$. The general form of the equation is given by [2]

$$\left|\frac{\mathrm{d}\varphi(x)}{\mathrm{d}x}\right| = m \cdot \frac{\rho(x)}{\rho(\varphi(x))} \quad (m = 2, 3, \ldots), \tag{1.3}$$

which is a differential equation of $\varphi(x)$ for given $\rho(x)$. Eq. (1.2) is the special case of $X = \varphi(x)$, $\rho(x) = 1/\pi\sqrt{1-x^2}$ and $m = 2$.

For the concreteness, let us show more examples of such one-dimensional chaotic maps. The first one is for the measure

$$\rho(x) = 1 - |x| \quad \bigl(|x| \leqslant 1\bigr) \tag{1.4}$$

and the map is $(m = 2)$

$$X = \varphi(x) = \begin{cases} -1 + \sqrt{2|x|(2-|x|)} & (|x| \leqslant x_0) \\ 1 - \sqrt{2}(1-|x|) & (x_0 < |x| \leqslant 1) \end{cases} \tag{1.5}$$

where $x_0 = 1 - \frac{1}{\sqrt{2}}$. The second example is for the measure

$$\rho(x) = \frac{1}{4\sqrt{1-|x|}} \quad \bigl(|x| < 1\bigr) \tag{1.6}$$

and the map is $(m = 2)$

$$X = \varphi(x) = \begin{cases} -1 + 4\bigl(1 - \sqrt{1-|x|}\,\bigr)^2 & \bigl(|x| \leqslant \frac{3}{4}\bigr), \\ 1 + 4(|x| - 1) & \bigl(\frac{3}{4} < |x| < 1\bigr). \end{cases} \tag{1.7}$$

Both are proved by substituting them into (1.3) directly. The fact that these maps give the prescribed invariant measures respectively can be verified numerically by comparing calculated histograms with the measure functions.

It will be natural to consider the similar problem in higher dimensions. The purpose of the present Letter is to give such examples for two dimensions. Some examples have been already known, in which the Arnold map must be the best known example which is the square-to-square map [3]

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \bmod 1 \tag{1.8}$$

where $0 \leqslant x, y, X, Y < 1$. The invariant measure of Arnold map is uniform $\rho(x, y) = 1$ and a numerical computation gives such histogram as Fig. 1 by 10^8 iterations.

* Corresponding author.
E-mail address: sogo@sci.kitasato-u.ac.jp (K. Sogo).

0375-9601/$ – see front matter
doi:10.1016/j.physleta.2011.08.013

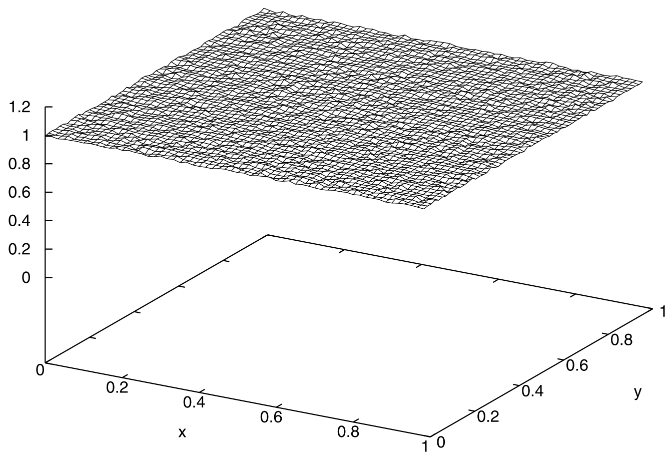

Fig. 1. The uniform distribution function $\rho(x, y) = 1$ for the Arnold map.

Another example is the three-term relation given by Grammaticos et al. (referred as GRV hereafter) [4]

$$x_{n+1} = \frac{3x_n - x_n^3 - x_{n-1}(1 - 3x_n^2)}{1 - 3x_n^2 + (3x_n - x_n^3)x_{n-1}}. \tag{1.9}$$

They considered this relation as two-dimensional map $(x_{n-1}, x_n) \to (x_n, x_{n+1})$ and found a chaotic behavior. Such behavior can be justified as follows. Since (1.9) is rewritten as

$$\frac{x_{n+1} + x_{n-1}}{1 - x_{n+1}x_{n-1}} = \frac{3x_n - x_n^3}{1 - 3x_n^2},$$

and this can be simplified by setting $x_n = \tan(\frac{\pi}{2}\theta_n)$ as

$$\theta_{n+1} + \theta_{n-1} = 3\theta_n \bmod [-1, 1]$$

or equivalently

$$\begin{pmatrix} \theta_n \\ \theta_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} \theta_{n-1} \\ \theta_n \end{pmatrix} \bmod I, \tag{1.10}$$

where $I = [-1, 1] \times [-1, 1]$. Note that the matrix is of Arnold type (the determinant $= 1$ and the trace > 2). Therefore the chaotic behavior of θ_n's induces that of x_n's. In the next section we will give another realization of GRV and show that the invariant measure is given by the product of Cauchy–Lorentz distribution function.

Two methods are proposed in the next section to obtain two-dimensional chaotic maps. One is the method to use already known one-dimensional maps in the polar coordinates. Another is the method to use two-dimensional Arnold map and some nonlinear transformations. Various examples are shown with corresponding invariant measures for each method. The last section is devoted to summary and remarks.

2. Two-dimensional maps and their invariant measures

2.1. Rotationally symmetric maps using one-dimensional maps

The first method is to use known one-dimensional maps. Then obtained two-dimensional maps will be shown to have rotationally symmetric invariant measures. The idea is to use the polar coordinates (r, ϕ), and to employ

$$R = f(r), \qquad \Phi = n\phi \bmod 2\pi \tag{2.1}$$

where $f(r)$ is a function derived from one-dimensional chaotic maps and n is typically set as $n = 2$. We can expect in advance that the procedure (2.1) gives necessary mixing property.

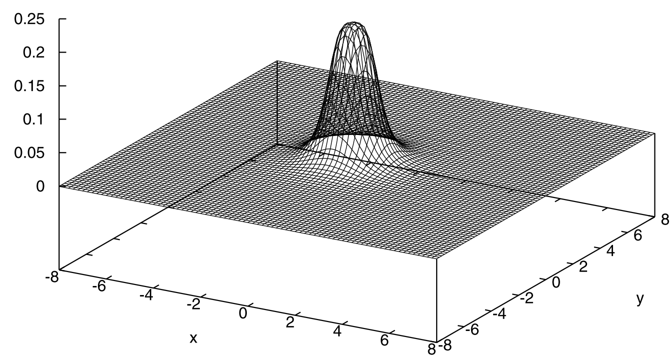

Fig. 2. The invariant measure $\rho(x, y) \propto 1/(1 + r^4)$.

2.1.1. Rotationally symmetric Lorentz map

The simplest one-dimensional map is probably

$$U = 2u \bmod 1 \quad (0 \leqslant u < 1) \tag{2.2}$$

whose invariant measure is uniform distribution $\rho(u) = 1$. Let us employ the polar coordinates $x = r\cos\phi$, $y = r\sin\phi$, $X = R\cos\Phi$, $Y = R\sin\Phi$, and consider the map

$$r^2 = \tan\left(\frac{\pi}{2}u\right), \qquad R^2 = \tan\left(\frac{\pi}{2}U\right). \tag{2.3}$$

Then for $0 \leqslant u < 1/2$,

$$R^2 = \tan\left(\frac{\pi}{2}U\right) = \tan\left(\frac{\pi}{2} \cdot 2u\right) = \frac{2\tan(\frac{\pi}{2}u)}{1 - \tan^2(\frac{\pi}{2}u)} = \frac{2r^2}{1 - r^4},$$

and for $1/2 \leqslant u < 1$,

$$R^2 = \tan\left(\frac{\pi}{2}U\right) = \tan\left(\frac{\pi}{2} \cdot (2u - 1)\right) = -\frac{1 - \tan^2(\frac{\pi}{2}u)}{2\tan(\frac{\pi}{2}u)} = \frac{r^4 - 1}{2r^2}.$$

We have, therefore, the direct map function

$$R = f(r) = \begin{cases} \sqrt{2r^2/(1 - r^4)} & (0 \leqslant r < 1), \\ \sqrt{(r^4 - 1)/2r^2} & (1 \leqslant r < \infty). \end{cases} \tag{2.4}$$

The corresponding invariant measure is derived as follows. Since

$$dx\,dy = r\,dr\,d\phi = \frac{\pi^2}{2} \cdot \left(1 + r^4\right) du \cdot \frac{d\phi}{2\pi},$$

we have

$$\frac{2}{\pi^2}\frac{1}{1 + r^4}\,dx\,dy = du \cdot \frac{d\phi}{2\pi}, \tag{2.5}$$

where it should be noted that the right-hand side (RHS) is normalized. The invariant measure is, therefore, rotationally symmetric and is given by

$$\rho(x, y) = \frac{A}{1 + r^4} \quad \left(r^2 = x^2 + y^2\right) \tag{2.6}$$

where we set $A = 2/\pi^2 = 0.202642$. This distribution function is shown in Fig. 2, and numerical computation gives the same figure.