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Solution of problems in calculus of variations via He's variational iteration method

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Abstract

In the modeling of a large class of problems in science and engineering, the minimization of a functional is appeared. Finding the solution of these problems needs to solve the corresponding ordinary differential equations which are generally nonlinear. In recent years He's variational iteration method has been attracted a lot of attention of the researchers for solving nonlinear problems. This method finds the solution of the problem without any discretization of the equation. Since this method gives a closed form solution of the problem and avoids the round off errors, it can be considered as an efficient method for solving various kinds of problems. In this research He's variational iteration method will be employed for solving some problems in calculus of variations. Some examples are presented to show the efficiency of the proposed technique.

Keywords: Calculus of variations; Euler–Lagrange equation; He's variational iteraction method; Symbolic computations; Extremum problem

1. Introduction

In this work we consider He's variational iteration method as a well known method for finding both analytic and approximate solutions of differential equations. The efficiency of this method for solving various types of problems is shown for example in [1–9]. Employing this technique the exact solution of a linear problem can be obtained by doing only one iteration step.

This method is used for solving autonomous ordinary differential systems in [2]. Application of this method to Helmholtz equation is investigated in [10]. This method is used for solving Burgers' and coupled Burgers' equations in [11]. In [11] the applications of the present method to coupled Schrödinger–KdV equations and shallow water equation are provided. Also the use of this method for solving linear fractional partial differential equations arising from fluid mechanics is discussed in [12]. Other recent works in this field are found in [13–17].

In the large number of problems arising in analysis, mechanics, geometry, it is necessary to determine the maximal and minimal of a certain functional. Because of the important role of this subject in science and engineering, considerable attention has been received on these kinds of problems. Such problems are called *variational problems* [18].

One well known method for solving variational problems is direct method. In this technique the variational problem is regarded as a limiting case of a finite number of variables. This extremum problem of a function of a finite number of variables is solved by ordinary methods, then a passage of limit yields the solution of the appropriate variational problem [19]. The direct method of Ritz and Galerkin has been investigated for solving variational problems in [19,20]. Using Walsh series method a piecewise constant solution is obtained for variational methods [21]. Some orthogonal polynomials are applied on variational problems to find continuous solutions for these problems [22–24]. Also Fourier series and Taylor series are applied to variational problems, respectively in [25] and [26] to find a continuous solution for these kinds of problems.

More historical comments about variational problems are found in [19,20].

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The organization of the rest of this Letter is as follows:

The well known He's variational iteration method is reviewed in Section 2. In Section 3, we introduce the general form of problems in calculus of variations, and their relations with ordinary differential equations are highlighted. To present a clear overview of the procedure, we select several examples in Section 3.1–3.4. A conclusion is presented in Section 4.

2. Variational iteration method

In this technique, the problem is initially approximated with possible unknowns. Then a correction functional is constructed by a general Lagrange multiplier, which can be identified optimally via the variational theory [3]. In this method the problem is considered as

$$Ly + Ny = g(x), \tag{2.1}$$

where L is a linear operator, and N is a nonlinear operator, $g(x)$ is an inhomogeneous term. Using the variational iteration method, the following correct functional is considered

$$y_{n+1} = y_n + \int_0^x \lambda\big(Ly_n(s) + N\tilde{y}_n(s) - g(s)\big)\,ds, \tag{2.2}$$

where λ is Lagrange multiplier [9], the subscript n denotes the nth approximation, $\tilde{y}_n$ is considered as a restricted variation i.e. $\delta\tilde{y}_n = 0$ [4–6]. Taking the variation from both sides of the correct functional with respect to y_n and imposing $\delta y_{n+1} = 0$, the stationary conditions are obtained. Using the stationary conditions the optimal value of the λ can be identified.

Since this procedure avoids the discretization of the problem, it is possible to find the closed form solution without any round off error. The use of symbolic computation is necessary for finding the iterations.

In the case of m equations, we rewrite equations in the form

$$L_i(y_i) + N_i(y_1, \ldots, y_m) = g_i(x), \quad i = 1, \ldots, m, \tag{2.3}$$

where L_i is linear with respect to y_i, and N_i is the nonlinear part of the ith equation. In this case the correct functionals are made as

$$y_{i(n+1)} = y_{in} + \int_0^x \lambda_i\big(L_i\big(y_{in}(s)\big) + N\big(\tilde{y}_{1n}(s), \ldots, \tilde{y}_{mn}(s)\big) - g(s)\big)\,ds, \tag{2.4}$$

and the optimal values of λ_i, $i = 1, \ldots, m$ are obtained by taking the variation from both sides of the correct functionals and finding stationary conditions using

$$\delta y_{i(n+1)} = 0, \quad i = 1, \ldots, m.$$

3. Statement of the problem

The simplest form of a variational problem can be considered as

$$v\big[y(x)\big] = \int_{x_0}^{x_1} F\big(x, y(x), y'(x)\big)\,dx, \tag{3.1}$$

where v is the functional that its extremum must be found. To find the extreme value of v, the boundary points of the admissible curves are known in the following form

$$y(x_0) = \alpha, \qquad y(x_1) = \beta. \tag{3.2}$$

The necessary condition for the solution of the problem (3.1) is to satisfy the Euler–Lagrange equation

$$F_y - \frac{d}{dx}F_{y'} = 0, \tag{3.3}$$

with boundary conditions given in (3.2). The boundary value problem (3.3) does not always have a solution and if the solution exists, it may not be unique. Note that in many variational problems the existence of a solution is obvious from the physical or geometrical meaning of the problem and if the solution of Euler's equation satisfies the boundary conditions, it is unique. Also this unique extremal will be the solution of the given variational problem [19].

The general form of the variational problem (3.1) is

$$v[y_1, y_2, \ldots, y_n] = \int_{x_0}^{x_1} F(x, y_1, y_2, \ldots, y_n, y_1', y_2', \ldots, y_n')\,dx, \tag{3.4}$$

with the given boundary conditions for all functions

$$y_1(x_0) = \alpha_1, \quad y_2(x_0) = \alpha_2, \quad \ldots, \quad y_n(x_0) = \alpha_n, \tag{3.5}$$

$$y_1(x_1) = \beta_1, \quad y_2(x_1) = \beta_2, \quad \ldots, \quad y_n(x_1) = \beta_n. \tag{3.6}$$

Here the necessary condition for the extremum of the functional (3.4) is to satisfy the following system of second-order differential equations

$$F_{y_i} - \frac{d}{dx}F_{y_i'} = 0, \quad i = 1, 2, \ldots, n, \tag{3.7}$$

with boundary conditions given in (3.5) and (3.6).

The Euler–Lagrange equation is generally nonlinear. In this Letter we apply the variational iteration method for solving Euler–Lagrange equations which arise from problems in calculus of variations. It is shown that this scheme is efficient for solving these kinds of problems.

3.1. Example 1

As an elementary example we consider the following variational problem

$$\min v = \int_0^1 \big(y(x) + y'(x) - 4\exp(3x)\big)^2\,dx, \tag{3.8}$$

with given boundary conditions

$$y(0) = 1, \qquad y(1) = e^3. \tag{3.9}$$

The corresponding Euler–Lagrange equation is

$$y'' - y - 8\exp(3x) = 0, \tag{3.10}$$