Available online at www.sciencedirect.com

ScienceDirect

Physics Letters A 362 (2007) 430–434

PHYSICS LETTERS A

www.elsevier.com/locate/pla

Ultradiscrete Miura transformation

S. Kubo [a], S. Isojima [b,*], M. Murata [c], J. Satsuma [b]

[a] Statistics Bureau, Ministry of Internal Affairs and Communications, 19-1 Wakamatsu-cho, Shinjuku-ku, Tokyo 162-8668, Japan
[b] Department of Physics and Mathematics, Aoyama Gakuin University, 5-10-1 Fuchinobe, Sagamihara-shi, Kanagawa 229-8558, Japan
[c] Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan

Received 16 July 2006; accepted 18 October 2006

Available online 27 October 2006

Communicated by A.P. Fordy

Abstract

An ultradiscrete analogue of the Miura transformation is constructed through the bilinear form of the discrete KdV and modified KdV equations. This transformation maps solutions of the 'box and ball system with a carrier' to those of the 'box and ball system'. Explicit examples of solutions are also discussed.

PACS: 02.30.Ik; 05.45.Yv

Keywords: Soliton; Cellular automaton; KdV equation; Miura transformation

1. Introduction

It is well known that solutions of the modified Korteweg–de Vries (mKdV) equation with a parameter ζ,

$$\frac{\partial w}{\partial s} - 6(w^2+\zeta^2)\frac{\partial w}{\partial x} + \frac{\partial^3 w}{\partial x^3} = 0 \tag{1}$$

are transformed to those of the Korteweg–de Vries (KdV) equation

$$\frac{\partial v}{\partial s} + 6v\frac{\partial v}{\partial x} + \frac{\partial^3 v}{\partial x^3} = 0 \tag{2}$$

by the Miura transformation [1]

$$v = \pm\frac{\partial w}{\partial x} - w^2 - \zeta^2. \tag{3}$$

This transformation was a key relation in discovering the inverse scattering method for the KdV equation [2]. Since the discovery, the soliton theory has developed to a considerable extent. The full discrete analogue of this transformation was discussed by Nijhoff and Capel [3]. It is to be noted that Hirota implicitly gave the discrete Miura transformation in the context of the bilinear formalization [4].

Takahashi and one of the authors (JS) proposed a soliton cellular automaton [5], which is now often called a 'box and ball system' (BBS) [6]. A remarkable feature of this cellular automaton is that any state consists only of solitary waves and all the waves interact as solitons. A direct relationship between the KdV equation and the BBS is clarified by the procedure of ultradiscretization.

* Corresponding author.
E-mail address: isojima@gem.aoyama.ac.jp (S. Isojima).

0375-9601/$ – see front matter
doi:10.1016/j.physleta.2006.10.040

It is shown in [7] that the discrete Lotka–Volterra equation, which is a discrete analogue of the KdV equation [8], is transformed to the soliton cellular automaton or the BBS by a limiting procedure and variable transformations. Interaction state of solitary waves in the BBS is understood as a limit of the N-soliton solution of the KdV equation. Takahashi and Matsukidaira presented an ultradiscretized mKdV equation and showed that the equation is reduced to a 'box and ball system with a carrier' (BBSC) [9]. The relationship between solutions of the discrete mKdV equation and those of the BBSC is clarified recently in [10]. From both theoretical and applied points of view, it is an interesting problem to construct an ultradiscrete analogue of the Miura transformation which transforms solutions of the BBSC to those of the BBS.

In this Letter, we present an ultradiscrete analogue of the Miura transformation and discuss its property. In Section 2, we give discrete analogues of the KdV equation, the mKdV equation and the Miura transformation in the forms suitable for investigating the relationship to the BBS and the BBSC. Then these discrete systems are ultradiscretized in Section 3. The property of the ultradiscrete Miura transformation is discussed in detail also in this section. Finally we give concluding remarks in Section 4.

2. Discrete system

In order to study the relationship between the BBS and the BBSC, we first consider partial difference systems. We start from the bilinear form of the discrete mKdV equation [4,10,11],

$$(1+\mu) f_j^t g_{j-1}^{t-1} = f_j^{t-1} g_{j-1}^t + \mu f_{j-1}^t g_j^{t-1}, \tag{4a}$$

$$(1+\lambda) f_{j-1}^{t-1} g_j^t = f_j^{t-1} g_{j-1}^t + \lambda f_{j-1}^t g_j^{t-1}. \tag{4b}$$

The N-soliton solution of (4) is written as [10]

$$f_j^t = \sum_{\boldsymbol{\nu}} \exp\left(\sum_{i=1}^N \nu_i \xi_i + \sum_{i<i'}^N \nu_i \nu_{i'} a_{ii'}\right), \tag{5a}$$

$$g_j^t = \sum_{\boldsymbol{\nu}} \exp\left(\sum_{i=1}^N \nu_i (\xi_i - c_i) + \sum_{i<i'}^N \nu_i \nu_{i'} a_{ii'}\right) \tag{5b}$$

with

$$\xi_i = k_i j + \omega_i t + \xi_i^{(0)}, \tag{6a}$$

$$e^{k_i} = \frac{e^{c_i}(1+\lambda) + 1 + 1/\mu}{e^{c_i}(1+1/\mu) + 1 + \lambda}, \qquad e^{\omega_i} = \frac{e^{c_i}(1+1/\lambda) + 1 + \mu}{e^{c_i}(1+\mu) + 1 + 1/\lambda}, \tag{6b}$$

$$e^{a_{ii'}} = \frac{\sinh^2\{(c_i - c_{i'})/2\}}{\sinh^2\{(c_i + c_{i'})/2\}}, \tag{6c}$$

where c_i and $\xi_i^{(0)}$ are arbitrary parameters and $\boldsymbol{\nu} = (\nu_1, \nu_2, \ldots, \nu_N)$, $\nu_i \in \{0, 1\}$. If we define a dependent variable by

$$u_j^t = \frac{f_j^t g_{j-1}^t}{f_{j-1}^t g_j^t} \tag{7}$$

and set the boundary condition $\lim_{j\to-\infty} (f_j^t g_j^{t+1})/(f_j^{t+1} g_j^t) = 1$, we obtain from (4),

$$u_j^{t+1} = \frac{1+\lambda}{1+\mu} \frac{\mu + u_j^t \prod_{i=-\infty}^{j-1} (u_i^t / u_i^{t+1})}{u_j^t + \lambda \prod_{i=-\infty}^{j-1} (u_i^{t+1} / u_i^t)}. \tag{8}$$

This equation is directly related to the BBSC as shown in [10]. Note that the N-soliton solution (5) satisfies this boundary condition.

If we eliminate g_j^t from (4) we have the trilinear equation

$$(1+\gamma) f_{j-1}^{t-2} f_{j+1}^{t-1} f_j^t - \gamma f_j^{t-2} f_{j+1}^{t-1} f_{j-1}^t = (1+\gamma) f_j^{t-2} f_{j-1}^{t-1} f_{j+1}^t - \gamma f_{j+1}^{t-2} f_{j-1}^{t-1} f_j^t, \tag{9}$$

where γ is a parameter defined by $\gamma = (\mu - \lambda)/(1 + 2\lambda + \mu\lambda)$. By introducing a dependent variable

$$b_j^t = \frac{f_j^{t-1} f_{j-1}^t}{f_{j-1}^{t-1} f_j^t} \tag{10}$$

(9) is reduced to

$$\frac{1+\gamma}{b_{j+1}^t} + \gamma b_j^t = \frac{1+\gamma}{b_j^{t-1}} + \gamma b_{j+1}^{t-1}, \tag{11}$$