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Fisher information contains all HO-quantum-statistics already at the semiclassical level

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Abstract

We study here the difference between quantum statistical treatments and semiclassical ones, using as the main research tool a semiclassical, shift-invariant Fisher information measure built up with Husimi distributions. Its semiclassical character notwithstanding, this measure also contains abundant information of a purely quantal nature. Such a tool allows us to refine the celebrated Lieb bound for Wehrl entropies and to discover thermodynamic-like relations that involve the degree of delocalization. Fisher-related thermal uncertainty relations are developed and the degree of purity of canonical distributions, regarded as mixed states, is connected to this Fisher measure as well.

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1. Introduction

A quarter of century before Shannon, R.A. Fisher advanced a method to measure the information content of continuous, rather than digital inputs, using not the binary computer codes but rather the statistical distribution of classical probability theory [1,2]. Already in 1980 Wootters pointed out that Fisher's information measure (FIM) and quantum mechanics share a common formalism and both relate probabilities to the squares of continuous functions [3]. Since then, much interesting work has been devoted to the manifold physical FIM-applications. For examples (not an exhaustive list, of course), see, for instance, Refs. [1,4–7]).

The semiclassical approximation (SC) has had a long and distinguished history and remains today a very important weapon in the physics' armory. It is indeed an indispensable one in many areas of scientific endeavor. Also, it facilitates, in many circumstances, an intuitive understanding of the underlying physics that may remain hidden in expensive numerical solutions of Schrödinger's equation. Although the SC-approach is as old as quantum mechanics itself, it remains active, as reported, for example, in Refs. [8,9].

Our emphasis in this communication will be placed on the study of the differences between (i) statistical treatments of a purely quantal nature, on the one hand, and (ii) semiclassical ones, on the other one. We will show that these differences can be neatly expressed entirely in terms of a special version, to be called I_z, of Fisher's information measure: the so-called shift-invariant Fisher one [1], in this Letter associated to phase-space. Additionally I_z is a functional of a semiclassical distribution function, namely, the Husimi function $\mu(x, p)$. The phase-space measure I_z will be shown to help one (1) to refine

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the so-called Lieb-bound [10] and (2) to connect this refinement with the phenomenon of delocalization in phase-space. The latter can, of course, be visualized as information loss. I_z will also be related to an interesting semiclassical measure that was early introduced to characterize the same phenomenon: *the Wehrl entropy W* [10],

$$W = -k_B \langle \ln \mu \rangle_\mu, \tag{1}$$

for which Lieb established the above cited lower bound $W \geqslant 1$, which is a manifestation of the uncertainty principle [11]. k_B is the Boltzmann's constant. Henceforth we will set $k_B = 1$, for simplicity's sake.

In the wake of a discussion advanced in Ref. [12], *we will be mainly concerned with building "Husimi–Fisher" bridges*. It is well known that the oldest and most elaborate phase-space (PS) formulation of quantum mechanics is that of Wigner [13,14]. To every quantum state a PS function (the Wigner one) can be assigned. This PS function can, regrettably enough, assume negative values so that a probabilistic interpretation becomes questionable. Such limitation was overcome by Husimi [15] (among others). In terms of the concomitant Husimi probability distributions, quantum mechanics can be completely reformulated [16,17]. For a given Hamiltonian $\mathcal{H}$ and an inverse temperature $\beta = 1/k_B T$, the normalized Husimi phase-space distributions $\mu(x, p)$ are built up as phase space-(x, p)-expectation values of Gibbs statistical operator

$$\rho \propto \exp(-\beta\mathcal{H}), \tag{2}$$

with respect to a coherent state $|z\rangle$. In turn, $|z\rangle$ is, of course, the eigenvector of an annihilation operator a, i.e., [18],

$$a|z\rangle = z|z\rangle \tag{3}$$

with (i) $z = Ax + iBp$ and (ii) A, B two constants depending upon the nature of the operator a. Thus one writes

$$\mu(x, p) \equiv \mu(z) = \langle z|\rho|z\rangle. \tag{4}$$

For the particular case of the harmonic oscillator (HO) of frequency ω one has [18]

$$\mu(z) = \left(1 - e^{-\beta\hbar\omega}\right) e^{-(1-e^{-\beta\hbar\omega})|z|^2}. \tag{5}$$

If $\mu(z)$ is given, its associated, semiclassical (phase space)-shift-invariant I_z writes [18]

$$I_z = \int \frac{\mathrm{d}^2 z}{\pi} \mu(z) \left\{ \frac{1}{4A^2} \left(\frac{\partial \ln \mu(z)}{\partial x} \right)^2 + \frac{1}{2B^2} \left(\frac{\partial \ln \mu(z)}{\partial p} \right)^2 \right\}, \tag{6}$$

where the constants are introduced for the sake of dimensional balance [18]. In the HO case the above specializes to

$$I_z(HO) = 1 - e^{-\beta\hbar\omega}. \tag{7}$$

Our HO findings are of relevance given the overwhelming influence of harmonic vibrations in all walks of physics, chemistry, astronomy and biology. For an active professional-physicist, it goes without saying that the HO is much more than a *mere example*, being nowadays of particular interest for the dynamics of bosonic or fermionic atoms contained in magnetic traps [19–21], as well as for any system that exhibits an equidistant level spacing in the vicinity of the ground state, like nuclei or Luttinger liquids. Additionally, the marriage between special relativity and quantum mechanics, i.e., quantum field theory, is essentially based upon the harmonic approximation [22]. Thus, new HO-results should always be of interest.

The Letter is organized as follows. We exhibit some interesting properties of the HO-I_z in Section 2. Appropriately employing the semiclassical measure we uncover the rather surprising amount of purely quantum information that I_z carries. Thermodynamic-like relations are derived in Section 3. A survey on thermal uncertainty relations is introduced in Section 4 in order to show that, via Fisher, quantum and semiclassical uncertainties can be linked. We are able to establish some special connections between degrees of purity in Section 5 and, finally, some conclusions are drawn in Section 6.

2. Properties of the HO-semiclassical Fisher's measure

Given the form (7), and since the temperature lies between zero and infinity, the range of values of I_z is

$$0 \leqslant I_z \leqslant 1. \tag{8}$$

Now, we wish to compare semiclassical quantities with the well-known *quantal* HO-expressions for, respectively, the partition function Z, the entropy S, the mean energy U, the mean excitation energy E, the free energy $F = U - TS$, and the specific heat C, that write [23]

$$Z = \frac{e^{-\beta\hbar\omega/2}}{1 - e^{-\beta\hbar\omega}}, \tag{9}$$

$$S = \beta \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1} - \ln\left\{1 - e^{-\beta\hbar\omega}\right\}, \tag{10}$$

$$U = \frac{\hbar\omega}{2} + E, \tag{11}$$

$$E = \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1}, \tag{12}$$

$$F = \frac{\hbar\omega}{2} + T \ln\left\{1 - e^{-\beta\hbar\omega}\right\}, \tag{13}$$

$$C = \left(\frac{\beta\hbar\omega}{e^{\beta\hbar\omega} - 1} \right)^2 e^{\beta\hbar\omega}. \tag{14}$$

Eq. (7) entails, via Eqs. (9)–(14), that the *quantal* HO expressions for the most important thermodynamic quantities can be expressed in terms of the *semiclassical* measure I_z. For such end we define the semiclassical free energy

$$F_{\mathrm{sc}} = T \ln I_z, \tag{15}$$

which is the semiclassical contribution to the HO free-energy $F = \hbar\omega/2 + F_{\mathrm{sc}}$. The above thermodynamic quantities can be expressed as follows

$$Z = \frac{e^{-\beta\hbar\omega/2}}{I_z}, \tag{16}$$

$$E = \hbar\omega \frac{1 - I_z}{I_z}, \tag{17}$$