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New travelling wave solutions of different physical structures to generalized BBM equation

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Abstract

New ansatze that involve hyperbolic functions are introduced to handle the generalized Benjamin–Bona–Mahony (BBM) equation. The tanh–sech method and the sine–cosine method are applied as well. A set of new solitons, kinks, and periodic solutions is formally established. The work introduces useful guidance for other related nonlinear evolution equations.

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1. Introduction

This Letter is concerned with the generalized BBM equation given by

$$u_t + u_x + au^n u_x + u_{3x} = 0, \quad n \geqslant 1, \tag{1}$$

with constant parameter a and $u_{kx} = \frac{\partial^k}{\partial x^k}$. For $n = 1$, Eq. (1) reduces to the BBM equation or the regularized long-wave equation (RLW) [1–10]

$$u_t + u_x + auu_x + u_{3x} = 0, \tag{2}$$

that describes surface waves in water in certain regimes and was proposed as an alternative to the KdV equation. However, for $n = 2$, Eq. (1) reduces to the modified BBM equation

$$u_t + u_x + au^2 u_x + u_{3x} = 0. \tag{3}$$

A great deal of research work has been invested during the past decades for the study of the nonlinear dispersive equations such as KdV, Boussinesq, and BBM equations [5–11]. The main goal of these works was directed towards its analytical solutions to find travelling wave solutions. Several different approaches, such as Bäcklund transformation, a bilinear form, and a Lax pair have been used independently by which soliton and multi-soliton solutions are obtained. Ablowitz et al. [4] implemented the inverse scattering transform method to handle the nonlinear equations of physical significance where soliton solutions and rational solutions were developed. The tanh method, developed by Malfliet [9], and used in [10–20] among many others, is heavily used in the literature to handle nonlinear evolutions equations. Fan et al. [4] introduced a useful extended tanh method that combines the standard tanh method with the Riccati equation. The extension worked effectively in nonlinear models and is used by many researchers. Recently, a set of ansatze that involve trigonometric and hyperbolic functions is introduced in [19–22] to develop compacton-like and solitary patterns-like solutions for nonlinear evolution equations.

Many strategies will be pursued to achieve our goal. The first approach depends on new ansatze that involve hyperbolic functions. The second approach rests mainly on the sine–cosine method and the well-known tanh method developed by Malfliet [9]. The proposed schemes will be used to determine distinct solutions of distinct physical structures.

In what follows, we highlight the main features of the proposed methods briefly, because details can be found in [9–11] and in [19–22]. The power of the methods, that will be used,

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is its ease of use to determine shock or solitary type of solutions.

2. The methods

In this section we present three ansatze, the tanh–sech method, sinh–cosh method, and the sine–cosine method to handle nonlinear equations in general, and the generalized BBM equation in particular. These schemes can be used directly in a straightforward manner to determine the unknown parameters involved in each ansatz.

2.1. A sinh–cosh ansatz I

In this case we use a cosh ansatz of the form

$$u(x,t) = \left(\frac{\alpha}{(1+\lambda\cosh[\mu(x-ct)])}\right)^{\frac{1}{n}}, \quad n \geqslant 1, \tag{4}$$

and a sinh ansatz

$$u(x,t) = \left(\frac{\alpha}{(1+\lambda\sinh[\mu(x-ct)])}\right)^{\frac{1}{n}}, \quad n \geqslant 1. \tag{5}$$

It is to be noted that the proposed ansatze can be applied directly to the given equation to determine the parameters α, λ, and μ.

2.2. A sinh–cosh ansatz II

In this case we use a cosh ansatz of the form

$$u(x,t) = \left(\frac{\alpha}{(1+\lambda\cosh^2[\mu(x-ct)])}\right)^{\frac{1}{n}}, \quad n \geqslant 1, \tag{6}$$

and a sinh ansatz

$$u(x,t) = \left(\frac{\alpha}{(1+\lambda\sinh^2[\mu(x-ct)])}\right)^{\frac{1}{n}}, \quad n \geqslant 1. \tag{7}$$

As stated before, the proposed ansatze can be applied directly to the given equation to determine the parameters α, λ, and μ.

2.3. A sinh–cosh ansatz III

In this case we use a cosh ansatz of the form

$$u(x,t) = \frac{\alpha\cosh^2[\mu(x-ct)]}{(1+\lambda\cosh^2[\mu(x-ct)])^{\frac{1}{n}}}, \quad n \geqslant 1, \tag{8}$$

and a sinh ansatz

$$u(x,t) = \frac{\alpha\sinh^2[\mu(x-ct)]}{(1+\lambda\sinh^2[\mu(x-ct)])^{\frac{1}{n}}}, \quad n \geqslant 1. \tag{9}$$

It is to be noted that the proposed ansatze can be applied directly to the given equation to determine the parameters α, λ, μ and c.

2.4. The sine–cosine ansatz

The PDE in two independent variables

$$P(u, u_t, u_x, u_{xx}, u_{xxx}, \ldots) = 0, \tag{10}$$

can be transformed to a nonlinear ODE upon using a wave variable $\xi = (x-ct)$. Eq. (10) is then integrated as long as all terms contain derivatives where integration constants are considered zeros.

The solutions of many nonlinear equations can be expressed in the form

$$\begin{aligned} u(\xi) &= \lambda\cos^{\beta}(\mu\xi), \\ (u^n)'' &= -n^2\mu^2\beta^2\lambda^n\cos^{n\beta}(\mu\xi) \\ &\quad + n\mu^2\lambda^n\beta(n\beta-1)\cos^{n\beta-2}(\mu\xi), \end{aligned} \tag{11}$$

and

$$\begin{aligned} u(\xi) &= \lambda\sin^{\beta}(\mu\xi), \\ (u^n)'' &= -n^2\mu^2\beta^2\lambda^n\sin^{n\beta}(\mu\xi) \\ &\quad + n\mu^2\lambda^n\beta(n\beta-1)\sin^{n\beta-2}(\mu\xi). \end{aligned} \tag{12}$$

Substituting (11) or (12) into the reduced ODE gives a trigonometric equation of $\cos^R(\mu\xi)$ or $\sin^R(\mu\xi)$ terms. The parameters are then determined by first balancing the exponents of each pair of cosine or sine to determine R. We next collect all coefficients of the same power in $\cos^k(\mu\xi)$ or $\sin^k(\mu\xi)$, where these coefficients have to vanish.

2.5. The tanh–sech method

The tanh method is a powerful solution method for the computation of exact traveling wave solutions [9–11]. Various extension forms of the tanh method have been developed. First a power series in tanh was used as an ansatz to obtain analytical solutions of traveling wave type of certain nonlinear evolution equations.

To avoid complexity, Malfliet [9] had customized the tanh technique by introducing tanh as a new variable, since all derivatives of a tanh are represented by a tanh itself. A straightforward analysis can then be carried out so that the method will be applicable to a large class of equations.

The tanh method introduces a new independent variable

$$Y = \tanh(\mu\xi), \tag{13}$$

that leads to the change of derivatives:

$$\begin{aligned} \frac{d}{d\xi} &= \mu\left(1-Y^2\right)\frac{d}{dY}, \\ \frac{d^2}{d\xi^2} &= \mu^2\left(1-Y^2\right)\left(-2Y\frac{d}{dY} + \left(1-Y^2\right)\frac{d^2}{dY^2}\right). \end{aligned} \tag{14}$$

The solutions can be proposed as a finite power series in Y in the form

$$u(\mu\xi) = S(Y) = \sum_{k=0}^{M} a_k Y^k, \tag{15}$$