Available online at www.sciencedirect.com

SCIENCE DIRECT

Physics Letters A 355 (2006) 212–215

PHYSICS LETTERS A

www.elsevier.com/locate/pla

Envelope compactons and solitary patterns

Zhenya Yan

CCAST (World Laboratory), PO Box 8730, Beijing 100080, PR China

Key Laboratory of Mathematics Mechanization, Institute of Systems Science, AMSS, Chinese Academy of Sciences, Beijing 100080, PR China [1]

Received 8 February 2006; accepted 14 February 2006

Available online 23 February 2006

Communicated by R. Wu

Abstract

In this Letter, to understand the role of nonlinear dispersion in complex nonlinear wave equations, we introduce and study nonlinear Schrödinger equation with nonlinear dispersion (called NLS(m,n) equation): $iu_t + (u|u|^{n-1})_{xx} + \mu u|u|^{m-1} = 0$. As a consequence, we obtain some envelope compactons for NLS$^+(n,n)$ equation and envelope solitary patterns for NLS$^-(n,n)$ equation. Moreover, we also show that NLS$(m,1)$ equation, which is nonlinear wave equation with linear dispersion possess both envelope compactons and solitary patterns. Finally, some unusually local conservation laws are given for NLS$^+(n,n)$ equation and NLS$^-(n,n)$ equation, respectively.

Keywords: NLS(m,n) equation; Envelope compactons; Envelope solitary patterns; Conservation laws

1. Introduction

In 1993, Rosenau and Hyman [1] presented a new type of solitons, called compactons, which usually are described by powers of trigonometric functions, $\sin(x)$ and $\cos(x)$, and exist in nonlinear wave equations with nonlinear dispersion. For example, $K(m,n)$ equation

$$u_t + \mu\big(u^m\big)_x + \big(u^n\big)_{xxx} = 0, \tag{1.1}$$

when $\mu = 1$, (1.1) is referred to as the focusing $(+)$ branch, the focusing $(+)$ branch exhibits compact structures, while when $\mu = -1$, (1.1) is referred to as the defocussing $(-)$ branch, the defocussing $(-)$ branch exhibits solitary pattern structures. Rosenau found that while compactons are the essence of the focusing $(+)$ branch, spikes, peaks and cusps are the hallmark of the defocussing $(-)$ branch which also supports the motion of kinks. The defocussing branch was found to give rise to solitary patterns having infinite slopes or cusps [2–9].

Recently, we have shown that nonlinear dispersion is not necessary condition to possess compactons and solitary patterns for nonlinear wave equation. Many nonlinear wave equations with linear dispersion also admit these types of solutions [6–9]. In addition, Kevrekidis et al. [10] presented discrete compactons of discrete models, Klein–Gordon-type different–different equation

$$\ddot{u}_n = g(u_n)(u_{n+1} + u_{n-1}) + f(u_n). \tag{1.3}$$

In this Letter, to understand the role of nonlinear dispersion in complex nonlinear wave equations, we introduce and study nonlinear Schrödinger equation with nonlinear dispersion (called NLS(m,n) equation), described by

$$iu_t + \big(u|u|^{n-1}\big)_{xx} + \mu u|u|^{m-1} = 0, \tag{1.2}$$

where $\mu = \pm 1$. When $m = 3$, $n = 1$, NLS$(3,1)$ equation becomes the usual nonlinear Schrödinger (NLS) equation:

$$iu_t + u_{xx} + \mu u|u|^2 = 0. \tag{1.4}$$

With the aid of symbolic computation, we obtain some *envelope compactons and solitary patterns* of NLS(m,n) equation using some transformations. In addition, some conservation laws of two different branches: NLS$^+(n,n)$ equation and NLS$^-(n,n)$ equation are also given.

E-mail address: zyyan@mmrc.iss.ac.cn (Z. Yan).

[1] Mailing address.

0375-9601/$ – see front matter © 2006 Elsevier B.V. All rights reserved.
doi:10.1016/j.physleta.2006.02.032

2. Envelope compactons

When $\mu = 1$, (1.2) is referred to as the focusing $(+)$ branch and is signed as $\mathrm{NLS}^+(m,n)$ equation

$$iu_t + \left(u|u|^{n-1}\right)_{xx} + u|u|^{m-1} = 0. \tag{2.1}$$

We take the following transformation:

$$u(x,t) = U(\xi)\exp(i\sigma t), \quad \xi = kx, \tag{2.2}$$

where k, σ are constants to be determined.

The substitution of (2.2) into (2.1) yields the ODE

$$-\sigma U + k^2\left[n(n-1)U^{n-2}U'^2 + nU^{n-1}U''\right] + U^m = 0. \tag{2.3}$$

To study compactons of (2.3), we assume that (2.3) has the solution

$$U(\xi) = \begin{cases} A\cos^\beta(\xi), & |\xi| \leqslant \frac{\pi}{2}, \\ 0, & \text{otherwise}, \end{cases} \tag{2.4}$$

where A and β are constants to be determined. With the aid of symbolic computation, the substitution of (2.4) into (2.3) yields

$$\begin{aligned} &-\sigma A\cos^\beta(\xi) + A^m\cos^{m\beta}(\xi) + k^2 n\beta(n\beta-1)A^n\cos^{n\beta-2}(\xi) \\ &\quad - k^2n^2\beta^2A^n\cos^{n\beta}(\xi) = 0. \end{aligned} \tag{2.5}$$

Therefore, we have the following sets of equations:

Type 1. $n \neq 1$

$$\begin{cases} \beta = n\beta - 2, \\ m\beta = n\beta, \\ -\sigma A + k^2n\beta(n\beta-1)A^n = 0, \\ A^m - k^2n^2\beta^2A^n = 0. \end{cases}$$

Type 2. $n = 1$

$$\begin{cases} m\beta = \beta - 2, \\ -\sigma A - k^2\beta^2 A = 0, \\ A^m + k^2\beta(\beta-1)A = 0. \end{cases}$$

From the above two sets of equation, we have the following solutions:

$$\begin{aligned} &m = n, \qquad \beta = \frac{2}{n-1}, \\ &k = \frac{n-1}{2n}, \qquad A^{n-1} = \frac{2n\sigma}{n+1}, \end{aligned} \tag{2.6}$$

$$\begin{aligned} &n = 1, \qquad \beta = \frac{2}{1-m}, \\ &k = \frac{1-m}{2}\sqrt{-\sigma}, \qquad A^{m-1} = \frac{\sigma(m+1)}{2}. \end{aligned} \tag{2.7}$$

Therefore, it follows from (2.6) that $\mathrm{NLS}^+(n,n)$ equation (2.1) with $n > 1$ has the envelope compacton solution

$$u(x,t) = \begin{cases} \left[\frac{2n\sigma}{n+1}\cos^2\left(\frac{n-1}{2n}x\right)\right]^{1/(n-1)}e^{i\sigma t}, & \left|\frac{n-1}{2n}x\right| \leqslant \frac{\pi}{2}, \\ 0, & \text{otherwise}. \end{cases} \tag{2.8}$$

In addition, from (2.7), we have the envelope compacton solution of $\mathrm{NLS}^+(m,1)$ equation (2.1) with $m < 1$,

$$u(x,t) = \begin{cases} \left[\frac{2}{\sigma(m+1)}\cos^2\left(\frac{1-m}{2}\sqrt{-\sigma}x\right)\right]^{1/(1-m)}e^{i\sigma t}, \\ \quad \left|\frac{1-m}{2}\sqrt{-\sigma}x\right| \leqslant \frac{\pi}{2}, \\ 0, \quad \text{otherwise}. \end{cases} \tag{2.9}$$

Similarly, if we use another transformation for (2.3),

$$U(\xi) = \begin{cases} A\sin^\beta(\xi), & 0 \leqslant \xi \leqslant \pi, \\ 0, & \text{otherwise}, \end{cases} \tag{2.10}$$

then we have another envelope compacton solution of $\mathrm{NLS}^+(n,n)$ equation (2.1) with $n > 1$,

$$u(x,t) = \begin{cases} \left[\frac{2n\sigma}{n+1}\sin^2\left(\frac{n-1}{2n}x\right)\right]^{1/(n-1)}e^{i\sigma t}, \\ \quad 0 \leqslant \frac{n-1}{2n}x \leqslant \pi, \\ 0, \quad \text{otherwise} \end{cases} \tag{2.11}$$

and the envelope compacton solution of $\mathrm{NLS}^+(m,1)$ equation (2.1) with $m < 1$,

$$u(x,t) = \begin{cases} \left[\frac{2}{\sigma(m+1)}\sin^2\left(\frac{1-m}{2}\sqrt{-\sigma}x\right)\right]^{1/(1-m)}e^{i\sigma t}, \\ \quad 0 \leqslant \frac{1-m}{2}\sqrt{-\sigma}x \leqslant \pi, \\ 0, \quad \text{otherwise}. \end{cases} \tag{2.12}$$

Remark 1. When $n = 1$ and $m < 1$, $\mathrm{NLS}^+(m,1)$ equation is shown to possess envelope compactons, which implies that nonlinear dispersion is not necessary condition to admit compactons for nonlinear wave equations.

Remark 2. When $m = n = 1$, $\mathrm{NLS}^+(m,n)$ equation reduces to linear differential equation. When $m = n < 1$, solutions (2.8) and (2.11) are singular. When $n = 1$, $m > 1$, solutions (2.9) and (2.12) are also singular.

3. Envelope solitary patterns

When $\mu = -1$, (1.2) is referred to as the focusing $(-)$ branch and is signed as $\mathrm{NLS}^-(m,n)$ equation

$$iu_t + \left(u|u|^{n-1}\right)_{xx} - u|u|^{m-1} = 0. \tag{3.1}$$

We take the following transformation:

$$u(x,t) = U(\xi)\exp(i\sigma t), \quad \xi = kx, \tag{3.2}$$

where k, σ are constants to be determined.

The substitution of (3.2) into (3.1) yields the ODE

$$-\sigma U + k^2\left(n(n-1)U^{n-2}U'^2 + nU^{n-1}U''\right) - U^m = 0. \tag{3.3}$$

To study solitary patterns of (3.3), we assume that (3.3) has the solutions

$$U(\xi) = A\sinh^\beta(\xi), \tag{3.4}$$

$$U(\xi) = A\cosh^\beta(\xi), \tag{3.5}$$

where A and β are constants to be determined. With the aid of symbolic computation, the substitution of (3.4) into (3.3) yields

$$\begin{aligned} &-\sigma A\sinh^\beta(\xi) - A^m\sinh^{m\beta}(\xi) \\ &\quad + k^2n\beta(n\beta-1)A^n\sinh^{n\beta-2}(\xi) \\ &\quad + k^2n^2\beta^2A^n\sinh^{n\beta}(\xi) = 0. \end{aligned} \tag{3.6}$$

Therefore, we have the following sets of equations: