Physics Letters A 374 (2010) 4664–4669

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

A conjecture on the use of quantum algebras in the treatment of discrete systems

O. Civitarese *, M. Reboiro

Department of Physics, University of La Plata, c.c. 67, 1900 La Plata, Argentina

ARTICLE INFO

Article history:
Received 16 June 2010
Received in revised form 17 September 2010
Accepted 22 September 2010
Available online 28 September 2010
Communicated by P.R. Holland

Keywords:
Squeezing
Spin–spin interactions
Spin–photon interactions
Quantum algebras

ABSTRACT

The interactions between atomic spin-states, and between them and an external radiation field, can be described in terms of quantum algebras by a trade-off of bosonic and fermionic degrees of freedom and q-deformed schemes. In this Letter we discuss the use of this concept concerning the calculation of a spin observable, like the spin squeezing.

1. Introduction

Recent developments in quantum optics and in quantum computing have motivated several studies of spin systems and their interactions [1,2]. The interest in these systems is linked to potential applications to the field of quantum devices [3,4]. In previous publications we have addressed some questions concerning the response of spin systems to the interaction with radiation [5–7], as well as the response of spin-interacting arrays, like spin chains [8,9]. As discussed there, one of the main difficulties associated to the calculation of spin-squeezing [10,11] is the large dimensionality of the space of configurations needed to built up the spin density matrix. Among different algebraic methods, quantum groups have been, successfully, applied to the study of spin-chains, for a review see, for instance [12]. Also, the use of quantum algebras to treat fermion and boson systems was presented in [13], in dealing with the Dicke model [14]. In this Letter we conjecture about the equivalence of a Hamiltonian which describes spin–spin and spin–radiation interactions [16,15,17,18], and an effective q-deformed Hamiltonian which contains only spin–spin interactions. The conjecture is based on the replacement of the spin–radiation interaction-term by a linear term which is written as a function of the generators of the $su_q(2)$ or $su_q(3)$ algebras. This replacement leads to a purely fermion Hamiltonian with almost the same spectrum and eigenfunctions of the initial fermion–boson Hamiltonian [19]. Here, we shall extend on this notion by exploring the consequences of our conjecture when applied to a system of atoms with two and three levels interacting with a radiation field. The notion that a class of Hamiltonian, with interaction terms between fermions and bosons, may be reduced to effective forms with simpler interactions was explored in Ref. [20]. Therein, the method of small rotations, which may eliminate sectors of the Hamiltonian which commute with the generators of the rotations, was discussed, particularly, for Hamiltonians describing atomic levels interacting with boson modes. The method of Klimov and Sanchez-Soto [20] is making use of the separation of sectors of a given Hamiltonian by projecting them into the subspaces of adequate integrals of motion. Then, by the proper choice of rotation operators one may transform sectors of the Hamiltonian and replace terms of it by effective ones where the absorption of some degrees of freedom is realized by the rotations. The method is mathematically sound and it offers a valid alternative to the path which we are proposing here, that is to elaborate on the deformation of the algebras as a way to eliminate boson degrees of freedom from the original fermion–boson Hamiltonians. In this Letter we concentrate on this procedure as a conjecture and explore its consequences. Further developments will certainly be devoted to establish the link with the method of Klimov and Sanchez-Soto [21]. The formalism is presented in Section 2, where we introduce the essentials of the proposed mapping and elaborate on the relevant symmetries. In Section 3 we show the results of the calculations which we have performed to support the conjecture. The conclusions are drawn in Section 4.

2. Formalism

In this section we shall present the essentials of the formalism for two leading cases, that is two- and three-level atoms interact-

* Corresponding author.
E-mail address: civitare@fisica.unlp.edu.ar (O. Civitarese).

0375-9601/$ – see front matter
doi:10.1016/j.physleta.2010.09.045

ing with radiation. As it will be explained below, these two cases involve the deformation of the $su(2)$ and $su(3)$ algebras, respectively, and they have been taken as test-cases.

2.1. Two-level atoms

We shall consider a system of N atoms, each of them having two spin-states and interacting with photons. A physical realization of this system would be the excitation of two-level atoms in a cavity by an incoming photon [18,22].

We write the Hamiltonian of the system as

$$H = \omega_f S_z + \lambda \sum_{\substack{i,j=1 \\ i\neq j}}^{N} \left(S_+^{(j)} S_-^{(i)} + S_+^{(i)} S_-^{(j)}\right) + \omega_b \left(a^\dagger a + \frac{1}{2}\right) + \eta\left(a^\dagger S_- + S_+ a\right)$$
$$\approx \lambda \frac{N}{2}(N-1) + (\omega_f - \lambda) S_z - 2\lambda S_z^2 + \omega_b \left(a^\dagger a + \frac{1}{2}\right) + \eta\left(a^\dagger S_- + S_+ a\right), \quad (1)$$

where

$$S_+ = \sum_{j=1}^{N} S_+^{(j)}, \quad S_- = S_+^\dagger, \quad S_z = \sum_{j=1}^{N} S_z^{(j)} \quad (2)$$

are the ladder operators which rise (S_+), or lower (S_-) the states of the atoms, S_z is the number operator; the energy gap between the states of a given atom is ω_f ($\hbar = 1$), and ω_b is the energy of the external boson field (photons). The third term of the Hamiltonian is the free-photon field, and the last term is the interaction of the photons with the atoms. The operators S_+, S_- and S_z obey the commutation rules of the $su(2)$ algebra. The operators $S_\pm^{(j)}$ and $S_z^{(j)}$ are the generators of the j-th copy $su(2)_j$ of the algebra, where j is the atomic index.[1] The excitations of $k \leqslant N$ atoms is described by Dicke states [14]

$$|k\rangle_{at} = \sqrt{\frac{(N-k)!}{N!k!}} (S_+)^k |0\rangle. \quad (3)$$

Since the Hamiltonian of Eq. (1) contains boson degrees of freedom, the state which represents l photons is written as the number state

$$|l\rangle_{ph} = \frac{1}{\sqrt{l!}} {a^\dagger}^l |0\rangle. \quad (4)$$

The Hamiltonian of Eq. (1) commutes with the operator

$$\hat{L} = a^\dagger a + S_z + \frac{N}{2}. \quad (5)$$

In terms of $\hat{L}$, the Hamiltonian of Eq. (1) is written as

$$H = \lambda \frac{N}{2}(N-1) + \omega_b \hat{L} - \omega_b \frac{1}{2}(N-1) + (\omega_f - \omega_b - \lambda)\tilde{S}_z - 2\lambda \tilde{S}_z^2 + \eta\left(a^\dagger S_- + S_+ a\right). \quad (6)$$

The basis of Eq. (5) may be labeled by the eigenvalues $\hat{L}$ of the operator L, such that

$$\hat{L}|L,k\rangle = L|L,k\rangle, \quad (7)$$

where $L = l + k$. We shall then express the wave function of the photons and atoms as

$$|L,k\rangle = |L-k\rangle_{ph} \otimes |k\rangle_{at}. \quad (8)$$

In each L-subspace, the eigenvalues and eigenvectors of H, of Eq. (1), can readily be obtained. With the set of these exact solutions of H, the time evolution of the expectation value of a given operator O is written

$$\langle O(t)\rangle = \mathrm{Tr}\big(\rho(t) O\big) = \sum_{\alpha,\beta} \langle\beta|I\rangle\langle I|\alpha\rangle\langle\alpha|O|\beta\rangle e^{-i(E_\alpha - E_\beta)t}. \quad (9)$$

In the above equation $\rho(t)$ is the density operator $\rho(t) = U^\dagger(t)\rho(0)U(t)$, being $\rho(0) = |I\rangle\langle I|$; the state $|I\rangle$ is the initial state of the system, $\{E_\alpha\}$ and $\{|\alpha\rangle\}$ are the α-th eigenvalue and eigenvector of the Hamiltonian, and $U(t) = \exp(-iHt)$ is the evolution operator.

2.1.1. The conjecture about an effective $su_q(2)$ Hamiltonian

The quantum algebra $su_q(2)$ is a Hopf algebra deformation of $su(2)$ [23,24] whose generators are $\tilde{S}_\pm$ and $\tilde{S}_z$, which obey the commutation rules

$$[\tilde{S}_z, \tilde{S}_\pm] = \pm\tilde{S}_\pm, \qquad [\tilde{S}_+, \tilde{S}_-] = [2\tilde{S}_z]_q. \quad (10)$$

The q-analogue $[x]_q$ of a given object x (a c-number or an operator) is defined by[2]

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}. \quad (11)$$

The $su(2)$ algebra is recovered from Eq. (10) in the limit $q \to 1$. When q is not a root of unity, the irreducible representations of $su_q(2)$ are obtained as a straightforward generalization of those of $su(2)$ [24].

To the starting Hamiltonian (1) we assign, by construction, the effective Hamiltonian

$$H_{(L,q)} = \lambda \frac{N}{2}(N-1) + \omega_b L - \omega_b \frac{1}{2}(N-1) + (\omega_f - \omega_b - \lambda)\tilde{S}_z - 2\lambda\tilde{S}_z^2 + \chi(L,q)(\tilde{S}_+ + \tilde{S}_-) \quad (12)$$

where $\chi(L,q)$ is a scalar function

$$\chi(L,q) = \eta\sqrt{\frac{(L-k_m)(k_m+1)(N-k_m)}{[k_m+1]_q[N-k_m]_q}},$$
$$k_m = \frac{1}{3}(-1+L+N) - \frac{1}{3}\sqrt{1+L+L^2+N-NL+N^2}, \quad (13)$$

and $H_{(L,q)}$ will be realized in an $su_q(2)$ irreducible representation with the same dimension as the L-subspace (7). Note that, the Hamiltonians H of Eq. (1) and $H_{(L,q)}$ of Eq. (12) are different, since the latter does not have boson degrees of freedom, and in the limit $q \to 1$ $H_{(L,q)} \neq H$. The main result of this procedure is

[1] The tensor product $\prod_{j=1}^{N} su(2)_j$ is the carrier space for the representations of the fermion (atomic) sector of the Hamiltonian.

[2] We shall use q or z ($q = e^z$) as the deformation parameter, and we shall assume that q is real.