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Exponential stability of delayed Hopfield neural networks with various activation functions and polytopic uncertainties

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ABSTRACT

This Letter deals with the problem of exponential stability for a class of delayed Hopfield neural networks. Based on augmented parameter-dependent Lyapunov–Krasovskii functionals, new delay-dependent conditions for the global exponential stability are obtained for two cases of time-varying delays: the delays are differentiable and have an upper bound of the delay-derivatives, and the delays are bounded but not necessary to be differentiable. The conditions are presented in terms of linear matrix inequalities, which allow to compute simultaneously two bounds that characterize the exponential stability rate of the solution. Numerical examples are included to illustrate the effectiveness of our results.

1. Introduction

The delayed Hopfield neural networks (DHNNs) have been extensively studied due to their useful applications in image processing, especially in classification of patterns, associative memories, and other areas [8,9,15]. Stability analysis of DHNNs is a prerequisite for their successful applications and has been extensively explored with various stability criteria, see, e.g., [1,3,11,12,14,19] and the references therein. In conducting a periodicity or stability analysis of a neural network, the conditions to be imposed on the neural network are determined by the characteristics of various activation functions as well as network parameters. When neural networks are designed for problem solving, it is desirable for their activation functions to be general. To facilitate the design of neural networks, it is important that the neural networks with various activation functions and time-varying delays are studied. The generalization of activation functions will provide a wider scope for neural network designs and applications. The results of [4] present algebraic criteria for ascertaining global exponential periodicity and exponential stability of a class of recurrent neural networks with various activation functions are derived by using the comparison principle and the theory of monotone operator. On the other hand, a model of neural networks have arisen for signal, fuzzy systems, where the state-space data are subjected to affine uncertainties and belong to the polytope and the rate of change of the state depends not only on the current state of the systems but also its state at some times in the past [5,6,10,13,17]. Theoretically, stability analysis of the systems with time-varying delays is more complicated, especially for the case where the system matrices belong to some convex polytope. In this case, the parameter-dependent Lyapunov–Krasovskii functionals are constructed as the convex combination of a set of functions assures the robust stability of the nominal systems and the stability conditions must be solved upon a grid on the parameter space, which results in testing a finite number of linear matrix inequalities (LMIs) [2,6,7,16,18].

In this Letter, we investigate exponential stability of a class of DHNNs. The delay neural networks in consideration are time-varying with various activation functions and polytopic uncertainties. The activation functions are assumed to be globally Lipschitz continuous. Firstly, for time-varying delays which are differentiable and the upper bound of the derivatives of the delays are less than one, a new approach is developed to obtain a sufficient condition, which guarantees the global exponential stability of an equilibrium point of such kind of delayed neural networks. Secondly, for time-varying delays which are bounded but unnecessarily differentiable, a sufficient condition is proposed by using Razumikhin stability theorem. We extend the results of [1,4,14,19] to the DHNNs with polytopic uncertainties. Based on the augmented parameter-dependent Lyapunov–Krasovskii functional, new delay-dependent conditions for the exponential stability are established in terms

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of LMIs, which allow to compute simultaneously two bounds that characterize the exponential stability rate of the solution and can be easily determined by utilizing Matlab's LMI Control Toolbox.

2. Preliminaries

The following notations will be used throughout this Letter. R^+ denotes the set of all real non-negative numbers; R^n denotes the n-dimensional space with the scalar product $\langle .,. \rangle$ and the vector norm $\|.\|$; $R^{n\times r}$ denotes the space of all matrices of $(n\times r)$-dimension. Matrix A is symmetric if $A = A^T$, where A^T denotes the transpose of A. I denotes the identity matrix; $\lambda(A)$ denotes the set of all eigenvalues of A; $\lambda_{\max}(A) = \max\{\mathrm{Re}\,\lambda: \lambda \in \lambda(A)\}$; $\lambda_{\min}(A) = \min\{\mathrm{Re}\,\lambda: \lambda \in \lambda(A)\}$; $C([a,b], R^n)$ denotes the set of all R^n-valued continuous functions on $[a,b]$; matrix A is semi-positive definite ($A \geqslant 0$) if $\langle Ax, x\rangle \geqslant 0$ for all $x \in R^n$; A is positive definite ($A > 0$) if $\langle Ax, x\rangle > 0$ for all $x \neq 0$; $A \geqslant B$ means $A - B \geqslant 0$. Let us denote $x_t := \{x(t+s),\ s \in [-h, 0]\}$ the segment of the trajectory $x(t)$ with the norm $\|x_t\| = \sup_{t\in[-h,0]} \|x(t+s)\|$.

Consider a DHNNs with polytopic type uncertainties of the form

$$\dot u(t) = -A(\xi)u(t) + B(\xi)\bar h\big(u(t)\big) + C(\xi)\bar g\big(u\big(t-h(t)\big)\big) + \mathcal{I},\quad t \in R^+,$$
$$u(t) = \phi(t),\quad t \in [-h, 0], \tag{2.1}$$

where $u(t) = [u_1(t), u_2(t), \ldots, u_n(t)]^T \in R^n$ is the state vector of the neural network, and n is the number of neurals, $\bar f(u(t)) = [\bar f_1(u_1(t)), \bar f_2(u_2(t)), \ldots, \bar f_n(u_n(t))]^T$, $\bar g(u(t)) = [\bar g_1(u_1(t)), \bar g_2(u_2(t)), \ldots, \bar g_n(u_n(t))]^T$ are the neural activation functions; $\mathcal{I} = [\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n]^T$ is the constant external input vector. The diagonal matrix $A(.)$ representing self-feedback term and the connection weighting matrices $B(.)$, $C(.)$ are uncertain but belong to a polytope Ω given by

$$\Omega = \left\{[A,B,C](\xi) := \sum_{i=1}^{p}\xi_i[A_i, B_i, C_i],\ \sum_{i=1}^{p}\xi_i = 1,\ \xi_i \geqslant 0\right\},$$

with vertices $\{A_i, B_i, C_i\}$, where $A_i = \mathrm{diag}(a_1^i, a_2^i, \ldots, a_n^i)$, $B_i = \{b_{ij}\}$, $C_i = \{c_{ij}\}$, $i, j = 1, \ldots, n$ are given constant matrices of appropriate dimensions, and $\xi_i (i = 1, \ldots, p)$ are time-invariant uncertainties. The initial functions $\phi(t) \in C([-h, 0], R^n)$ with the uniform norm $\|\phi\| = \max_{t\in[-h,0]} \|\phi(t)\|$. The time-varying delay function $h(t)$ satisfies either (A1) or (A2):

(A1) $0 \leqslant h(t) \leqslant h$, $\dot h(t) \leqslant \delta$, $\forall t \geqslant 0$,
(A2) $0 \leqslant h(t) \leqslant h$, $\forall t \geqslant 0$.

In the field of neural networks, a typical assumption is that the activation functions are the same and continuous, differentiable, monotonically increasing, and bounded. In this Letter, we assume the following assumption.

(H_1) The activation functions $\bar f(u)$, $\bar g(u)$ are bounded and global Lipschitz with the Lipschitz constants $f_i, g_i > 0$:

$$\big|\bar f_i(\xi_1) - \bar f_i(\xi_2)\big| \leqslant f_i|\xi_1 - \xi_2|,\quad i = 1,2,\ldots,n,\ \forall \xi_1, \xi_2 \in R$$
$$\big|\bar g_i(\xi_1) - \bar g_i(\xi_2)\big| \leqslant g_i|\xi_1 - \xi_2|,\quad i = 1,2,\ldots,n,\ \forall \xi_1, \xi_2 \in R. \tag{2.2}$$

As usual, vector $u^* = [u_1^*, u_2^*, \ldots, u_n^*]^T$ denotes an equilibrium point of system (2.1). It is well known that, under the above assumption, there is an equilibrium point u^* for (2.1), then by setting $x(t) = u(t) - u^*$, the system (2.1) can be transformed to the system

$$\dot x(t) = -A(\xi)x(t) + B(\xi)f\big(x(t)\big) + C(\xi)g\big(x\big(t-h(t)\big)\big),\quad t \in R^+,$$
$$x(t) = \phi(t),\quad t \in [-h, 0], \tag{2.3}$$

where

$$f(x) = \big[f_1(x_1), f_2(x_2), \ldots, f_n(x_n)\big]^T,$$
$$g(x) = \big[g_1(x_1), g_2(x_2), \ldots, g_n(x_n)\big]^T,$$
$$f_i(x_i) = \bar f_i\big(x_i + u_i^*\big) - \bar f_i\big(u_i^*\big),$$
$$g_i(x_i) = \bar g_i\big(x_i + u_i^*\big) - \bar g_i\big(u_i^*\big),\quad i = 1,2,\ldots,n.$$

Then, from the condition (2.2), we have $f_i(0) = 0$, $g_i(0) = 0$, $i = 1,2,\ldots,n$, and

$$\big|f_i(\xi_1) - f_i(\xi_2)\big| \leqslant f_i|\xi_1 - \xi_2|,\quad i = 1,2,\ldots,n,\ \forall \xi_1, \xi_2 \in R,$$
$$\big|g_i(\xi_1) - g_i(\xi_2)\big| \leqslant g_i|\xi_1 - \xi_2|,\quad i = 1,2,\ldots,n,\ \forall \xi_1, \xi_2 \in R. \tag{2.4}$$

Definition 2.1. Given $\beta > 0$, the zero solution of system (2.3) is β-exponentially stable if every solution $x(t, \phi)$ of the system satisfies

$$\exists N > 0:\ \big\|x(t,\phi)\big\| \leqslant N\|\phi\|e^{-\beta t},\quad \forall t \geqslant 0.$$

The following propositions are essential for the proofs in the subsequent section.

Proposition 2.1. *Let P, Q be matrices of appropriate dimensions and Q is symmetric positive definite. Then*

$$2\langle Py, x\rangle - \langle Qy, y\rangle \leqslant \big\langle PQ^{-1}P^T x, x\big\rangle,$$

for all (x, y).

The proof of the above proposition is easily derived from completing the square:

$$0 \leqslant \big\langle Q\big(y - Q^{-1}P^T x\big), y - Q^{-1}P^T x\big\rangle.$$

Proposition 2.2 *(Razumikhin stability theorem [5]). Consider the time delay system $\dot x(t) = f(t, x_t)$, $x(t) = \phi(t)$, $t \in [-h, 0]$. Assume that $u, v, w : R^+ \to R^+$ are nondecreasing, and $u(s)$, $v(s)$ are positive for $s \geqslant 0$, $v(0) = u(0) = 0$, and $q > 1$. If there is a function $V(t, x) : R^+ \times R^n \to R^+$ such that*

(i) *$u(\|x\|) \leqslant V(t, x) \leqslant v(\|x\|)$, $t \in R^+$, $x \in R^n$,*
(ii) *$\dot V(t, x(t)) \leqslant -w(\|x(t)\|)$ if $V(t+s, x(t+s)) \leqslant qV(t, x(t))$, $\forall s \in [-h, 0]$, $t \geqslant 0$,*

then the zero solution of system is asymptotically stable.

Proposition 2.3. *For real numbers $\xi_i \geqslant 0$, $i = 1,2,\ldots,p$, $\sum_{i=1}^p \xi_i = 1$, the following inequalities hold*

(i) $(p-1)\sum_{i=1}^p \xi_i^2 - 2\sum_{i=1}^{p-1}\sum_{j=i+1}^p \xi_i\xi_j \geqslant 0$,
(ii) $\sum_{i=1}^p \xi_i^2 + \sum_{i=1}^{p-1}\sum_{j=i+1}^p \xi_i\xi_j \geqslant \frac{1}{2} + \frac{1}{2n}$.

Proof. (i) is obvious from the relation

$$(p-1)\sum_{i=1}^{p}\xi_i^2 - 2\sum_{i=1}^{p-1}\sum_{j=i+1}^{p}\xi_i\xi_j = \sum_{i=1}^{p-1}\sum_{j=i+1}^{p}(\xi_i - \xi_j)^2.$$

To prove (ii), note that