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Quadratic forms for Feynman–Kac semigroups

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Abstract

Some problems in a stochastic setting often involve the need to evaluate the Feynman–Kac formula that follows from models described in terms of stochastic differential equations. Equivalent representations in terms of partial differential equations are also of interest, and these establish the well-known connection between probabilistic and deterministic formulations of these problems. In this Letter, this connection is studied in terms of the quadratic form associated with the Feynman–Kac semigroup. The probability measures that naturally arise in this approach, and thus define how Brownian motion is killed at a specified rate while exiting a set, are interpreted as a random time change of the original stochastic differential equation. Furthermore, since random time changes alter the diffusion coefficients in stochastic differential equations while Girsanov-type measure transformations alter their drift coefficients, their simultaneous use should lead to more tractable solutions for some classes of problems. For example, the minimization of some quadratic forms leads to solutions that satisfy certain partial differential equations and, therefore, the techniques discussed provide a variational approach for finding these solutions.

Keywords: Feynman–Kac; Semigroups; Resolvents; Variational problems; Quadratic forms; Stochastic differential equations; Partial differential equations; Cameron–Martin–Girsanov; Dirichlet problems

1. Introduction

The Feynman–Kac (FK) formula is familiar to most who have worked in the area of stochastic processes. Some applications of expressions related to this formula that have been used in detection, estimation and control theory include references [1–10]. A good reference that exploits the semigroup interpretation of this formula and whose first two chapters we often refer to is the text by Sznitman [11]. We shall also refer to the texts by Karatzas and Shreve [12], Durrett [13], Oksendal [14], Treves [15], Reed and Simon [16,17], and Freidlin and Wentzell [18] that provide excellent background material and problems for the topics we address here. In addition, some fairly recent papers related to FK semigroups that we have found very interesting and insightful are Kolokoltsov [19], Fitzsimmons and Kuwae [20], and Van Casteren [21].

The main contribution of this Letter is as follows. Although the random time change transformation to be presented is well known, we wish to emphasize its relation to the FK formula, its subsequent role in formulating boundary value problems described by partial differential equations (PDEs), and its use in models described by stochastic differential equations (SDEs); the well-known Girsanov measure transformation is also of importance in this regard. Using the existing theory of quadratic forms, we then present applications to problems whose solutions can be represented in terms of solving a variational problem. The overall approach, therefore, will be seen to unify optimization methodologies for some seemingly diverse problems and thereby will lead to tractable solutions.

The Letter is organized as follows. In Section 2, we begin with a two part mathematical formulation of the problem. Thus, in Section 2.1 we specify a probability space on which we define our model as a stochastic differential equation. We

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then specify the product probability measure that is used to define the FK formula and give the standard setup of how it relates to Brownian motion being killed as it exits a set at a particular rate. As we shall see, this can be interpreted as a random time change of the original SDE. Continuing in Section 2.2, we use the semigroup interpretation of the FK formula and introduce the quadratic form associated with it. Here, as in Sznitman [11] and Treves [15], we use these forms to express the solutions of PDEs of the Cauchy, Dirichlet and Schrödinger type as solutions to variational problems. Following this, we discuss specific applications in Section 3, where the first deals with the Cameron–Martin–Girsanov measure transformation, the second with Doob's h-transform, and the third relates Doob's h-transform to the theory of large deviations. Finally, we conclude in Section 4.

2. Mathematical formulation

2.1. Feynman–Kac formulas

We assume a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with a filtration $\{\mathcal{F}_{t\geqslant 0}\} \subseteq \mathcal{F}$, satisfying the usual conditions, that is generated by a stochastic process $x_t \in R^d$ that evolves according to the SDE

$$dx_t = b(x_t)\,dt + \sigma(x_t)\,dw_t, \quad x_0. \tag{2.1}$$

Here, $b(\cdot)$ and $\sigma(\cdot)$ are Lipshitz continuous functions satisfying linear growth conditions that guarantee unique solutions, and w_t is a d-dimensional Brownian motion with transition density

$$p(u,x,y) = (2\pi u)^{-d/2} \exp\left\{\frac{-(y-x)^2}{2u}\right\}, \quad u > 0. \tag{2.2}$$

We continue with the notation in Sznitman [11]. With $f \in C_0(R^d)$, the space of continuous functions tending to zero at infinity, and V a bounded continuous function on R^d, the FK semigroup is given by

$$R_t^V f(x) := E_x\left[f(w_t)\exp\left\{-\int_0^t V(w_s)\,ds\right\}\right], \quad t > 0.$$

This, however, can be generalized by constraining w_t to remain in some open subset U of R^d, whereby one gets

$$R_t^{U,V} f(x) := E_x\left[f(w_t)\exp\left\{-\int_0^t V(w_s)\,ds\right\},\ T_U > t\right], \tag{2.3}$$

where T_U, the exit time of Brownian motion from U, is defined as $T_U := \inf\{t \geqslant 0, w_t \notin U\}$.

As explained in Sznitman [11], such a generalization can present difficulties because of irregularity conditions on U (e.g., a lack of closure of $R_t^{U,V}$ when $f \in C_0(U)$), smoothness assumptions on V, and so on. Therefore, the space $C_0(R^d)$ is abandoned and replaced by $L^p(U,dx)$, $1 \leqslant p < \infty$, and the Kato space K_d and its localized version K_d^{loc} are defined as

$$K_d := \left\{f \text{ measurable: } \lim_{r\downarrow 0}\sup_x E_x\int_0^{H_r}|f|(w_s)\,ds = 0\right\},$$

$$K_d^{\mathrm{loc}} := \left\{f \text{ measurable: } \forall N \geqslant 1 \quad \lim_{r\downarrow 0}\sup_{|x|\leqslant N} E_x\int_0^{H_r}|f|(w_s)\,ds = 0\right\}, \tag{2.4}$$

where $H^r := \inf\{s \geqslant 0, |w_s - w_0| \geqslant r\}$. As a result, if the function $V(\cdot)$, the so-called potential, is such that $V_- := \max(-V,0) \in K_d$ and $V_+ := \max(V,0) \in K_d^{\mathrm{loc}}$, $f \in L^p(U,dx)$ is bounded and measurable, and U is a nonempty open subset of R^d, then $R_t^{U,V}$ in (2.3) defines a strongly continuous semigroup on $L^p(U,dx)$, $1 \leqslant p < \infty$, and, using the Brownian bridge measure, can be shown to satisfy

$$\begin{aligned} R_t^{U,V} f(x) &= E_x\left[f(w_t)E_{x,w_t}^t\left[\exp\left\{-\int_0^t V(w_s)\,ds\right\},\ T_U > t\right]\right] \\ &= \int f(y) r_{U,V}(t,x,y)\,dy, \quad t > 0, \quad x, y \in R^d, \end{aligned} \tag{2.5}$$

where its kernel $r_{U,V}$ is given by

$$r_{U,V}(t,x,y) := p(t,x,y)E_{x,y}^t\left[\exp\left\{-\int_0^t V(w_s)\,ds,\ T_U > t\right\}\right]. \tag{2.6}$$

Also, this semigroup is self-adjoint for $p = 2$ and, when $V \geqslant 0$, it is a contraction semigroup on $L^2(U,dx)$. These and other properties are stated and proved in Section 1.3 of Sznitman [11].

We now turn our attention to the product probability measure used to define the FK semigroup associated with Brownian motion being killed at a nonnegative rate $V \in K_d^{\mathrm{loc}}$ while exiting a nonempty connected open subset $U \subset R^d$. Toward this end, one defines a 'cemetery state' Δ and a canonical Brownian motion w_t such that there exists a random time τ, whereby $w_t \in U$ for $t \in [0,\tau)$ and $w_t = \Delta$ for $t \geqslant \tau$. Accordingly, the 'death time' is then defined as $\zeta := \inf\{s \geqslant 0, w_s = \Delta\}$. With the product space $C(R_+, R^d) \times R_+$, the product measure $\hat{P}$ is defined as $\hat{P}_z := P_z \otimes e^{-\xi}\,d\xi$, that is, the components of the product space are independent and are distributed, respectively, as Brownian motion starting from z and as an exponential random variable ξ with parameter 1. From here Sznitman [11] expresses the 'death time' as

$$\tau(w,\xi) = T_U(w) \wedge \inf\left\{s \geqslant 0, \int_0^s V(w_u)\,du \geqslant \xi\right\}, \tag{2.7}$$

the process

$$Y_t(w,\xi) = \begin{cases} w_t, & 0 \leqslant t \leqslant \tau(w,\xi), \\ \Delta, & t \geqslant \tau(w,\xi), \end{cases} \tag{2.8}$$