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Nonlinear waves in bubbly liquids with consideration for viscosity and heat transfer

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ABSTRACT

Nonlinear waves are studied in a mixture of liquid and gas bubbles. Influence of viscosity and heat transfer is taken into consideration on propagation of the pressure waves. Nonlinear evolution equations of the second and the third order for describing nonlinear waves in gas–liquid mixtures are derived. Exact solutions of these nonlinear evolution equations are found. Properties of nonlinear waves in a liquid with gas bubbles are discussed.

1. Introduction

A mixture of liquid and gas bubbles of the same size may be considered as an example of a classic nonlinear medium. In practice analysis of propagation of the pressure waves in a liquid with gas bubbles is important problem. Similar two-phase medium describes many processes in nature and engineering applications. In particular, such mathematical models are useful for studying dynamics of contrast agents in the blood flow at ultrasonic researches [1,2]. The literature on this subject deals with theoretical and experimental studies of the various aspects for propagation of the pressure waves in bubbly liquids.

The first analysis of a problem bubble dynamics was made by Rayleigh [3], who had solved the problem of the collapse of an empty cavity in a large mass of liquid. He also considered the problem of a gas-filled cavity under the assumption that the gas undergoes isothermal compression [4]. Based on the works of Rayleigh [3] and Foldy [5], van Wijngaarden [6] showed that in the case of one spatial dimension, the propagation of linear acoustic waves in isothermal bubbly liquids, wherein the bubbles are of uniform radius, is described by the linear partial differential equation of the fourth order [7].

The dynamic propagation of acoustic waves in a half-space filled with a viscous, bubbly liquid under van Wijngaarden linear theory was considered in the recent work [7]. However we know that there are solitary and periodic waves in a mixture of a liquid and gas bubbles and these waves can be described by nonlinear partial differential equations. As for examples of nonlinear differential equations to describe the pressure waves in bubbly liquids we can point out the Burgers equation [8–10], the Korteweg–de Vries equation [11–14], the Burgers–Korteweg–de Vries equation [14] and so on.

Many authors applied the numerical methods to study properties of the nonlinear pressure waves in a mixture of a liquid and gas bubbles. Nigmatulin and Khabeev [15] studied the heat transfer between a gas bubble and a liquid by means of the numerical approach. Later Aidagulov et al. [16] investigated the structure of shock waves in a liquid with gas bubbles with consideration for the heat transfer between gas and liquid. Oganyan in [17,18] tried to take into account the heat transfer between a gas bubble and a liquid to obtain the nonlinear evolution equations for the description of the pressure waves in a gas–liquid mixture. However, we have some different characteristic times of nonlinear waves in these processes and it turned out that the solution of this task is difficult problem.

The purpose of this work is to obtain more common nonlinear partial differential equations for describing the pressure waves in a mixture liquid and gas bubbles taking into consideration the viscosity of liquid and the heat transfer. We also look for exact solutions of these nonlinear differential equations to study the properties of nonlinear waves in a liquid with gas bubbles.

This Letter is organized as follows. System of equations for description of nonlinear waves in a mixture of liquid and gas bubbles taking into consideration for the heat transfer and the viscosity of

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liquid is given in Section 2. In Section 3 we obtain the basic nonlinear evolution equation to describe the pressure waves in liquid with gas bubbles. In Sections 4 and 5 we present nonlinear evolution equations of the second and the third order and search for exact solutions of these nonlinear differential equations.

2. System of equations for description of motion of liquid with gas bubbles with consideration for heat exchange and viscosity

Suppose that a mixture of a liquid and gas bubbles is homogeneous medium [19]. In this case for description of this mixture we use the averaged temperature, velocity, density and pressure. We also assume that the gas bubbles has the same size and the amount of bubbles in the mass unit is constant N. We take into account the processes of the heat transfer and viscosity on the boundary of bubble and liquid into account. We do not consider the processes of formation, destruction, and conglutination for the gas bubbles.

We have that the volume and the mass of gas in the unit of the mass mixture can be written as

$$V=\frac{4}{3}\pi R^3 N, \qquad X=V\rho_g,$$

where $R=R(x,t)$ is the bubble radius, $\rho_g=\rho_g(x,t)$ is the gas density. Here and later we believe that the subscript g corresponds to the gas phase and subscript l corresponds to the liquid phase.

We consider the long wavelength perturbations in a mixture of the liquid and gas bubbles assuming that characteristic length of waves of perturbation more than distance between bubbles. We also assume, that distance between bubbles much more than the averaged radius of a bubble.

We describe dynamics of a bubble using the Rayleigh–Lamb equation. We also take the equation of energy for a bubble and the state equation for the gas in a bubble into account. The system of equation for the description of the gas bubble takes the form [19,20]

$$\rho_l\left(RR_{tt}+\frac{3}{2}R_t^2+\frac{4\nu}{3R}R_t\right)=P_g-P, \tag{2.1}$$

$$P_{g,t}+\frac{3nP_g}{R}R_t+\frac{3\chi_g Nu\,(n-1)}{2R^2}(T_g-T_l)=0, \tag{2.2}$$

$$T_g=\frac{T_0P_g}{P_{g,0}}\left(\frac{R}{R_0}\right)^3, \tag{2.3}$$

where $P(x,t)$ is a pressure of a gas–liquid mixture, P_g is a gas pressure in a bubble, T_g and T_l are temperatures of liquid and gas accordingly, χ_g is a coefficient of the gas thermal conduction, Nu is the Nusselt number, n is a polytropic exponent, ν is the viscosity of a liquid.

The expression for the density of a mixture can be presented in the form [19]

$$\frac{1}{\rho}=\frac{1-X}{\rho_l}+V \;\Rightarrow\; \rho=\frac{\rho_l}{1-X+V\rho_l}. \tag{2.4}$$

Considering the small deviation of the bubble radius in comparison with the averaged radius of bubble, we have

$$R(x,t)=R_0+\eta(x,t), \quad R_0=\mathrm{const}, \quad \|\eta\|\ll R_0,$$
$$R(x,0)=R_0. \tag{2.5}$$

Assume that the liquid temperature is constant and equal to the initial value

$$T_l=T|_{t=0}=T_0, \quad T_0=\mathrm{const}.$$

At the initial moment, we also have

$$t=0:\quad P=P_g=P_0,\quad P_0=\mathrm{const},\qquad V=V_0=\frac{4}{3}\pi R_0^3N.$$

Substituting P_g and T_g from Eqs. (2.1) and (2.3) into Eq. (2.2) and taking relation (2.5) into account we have the pressure dependence of a mixture on the radius perturbation in the form

$$\begin{aligned}
&P-P_0+\frac{\eta}{R_0}P+\frac{3n\varkappa}{R_0}P\eta_t+\varkappa P_t+\frac{\rho_l(3R_0^2+4\nu\varkappa)}{3R_0}\eta_{tt}\\
&\quad+\frac{\rho_l(6R_0^2-4\nu\varkappa)}{3R_0^2}\eta\eta_{tt}+\frac{\rho_l(8\nu\varkappa(3n-1)+9R_0^2)}{6R_0^2}\eta_t^2\\
&\quad+\frac{4\nu\rho_l}{3R_0}\eta_t+\frac{2P_0}{R_0}\eta+\frac{3P_0}{R_0^2}\eta^2=0,\\
&\varkappa=\frac{2R_0^2P_0}{3\chi\,Nu\,(n-1)T_0}.
\end{aligned} \tag{2.6}$$

From Eq. (2.4) we also have the dependence ρ on η using formula (2.5)

$$\rho=\rho_0-\mu\eta+\mu_1\eta^2, \qquad \rho_0=\frac{\rho_l}{1-X+V_0\rho_l}, \tag{2.7}$$

$$\mu=\frac{3\rho_l^2V_0}{R_0(1-X+V_0\rho_l)^2}, \qquad \mu_1=\frac{6\rho_l^2V_0(2\rho_lV_0-1+X)}{R_0^2(1-X+\rho_lV_0)^3}. \tag{2.8}$$

We use the system of equations for description of the motion of a gas–liquid mixture flow in the form

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}=0, \qquad \rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\right)+\frac{\partial P}{\partial x}=0, \tag{2.9}$$

where $u=u(x,t)$ is a velocity of a flow of a gas–liquid mixture.

Eq. (2.9) together with Eqs. (2.6) and (2.8) can be applied for description of nonlinear waves in a gas–liquid medium.

Consider the linear case of the system of Eqs. (2.6), (2.8) and (2.9). Assuming, that pressure in a mixture is proportional to perturbation radius, we obtain the linear wave equation for the radius perturbations

$$\eta_{tt}=c_0^2\eta_{xx}, \quad c_0=\sqrt{\frac{3P_0}{\mu R_0}}. \tag{2.10}$$

Let us introduce the following dimensionless variables

$$t=\frac{l}{c_0}t', \qquad x=lx', \qquad u=c_0u',$$
$$\eta=R_0\eta', \qquad P=P_0P'+P_0,$$

where l is the characteristic length of wave.

Using the dimensionless variables the system of Eqs. (2.6), (2.8) and (2.9) can be reduced to the following (the primes of the variables are omitted)

$$\begin{aligned}
&\eta_t-\frac{\rho_0}{\mu R_0}u_x+u\eta_x+\eta u_x-\frac{2\mu_1R_0}{\mu}\eta\eta_t=0,\\
&-\frac{\rho_0}{\mu R_0}(u_t+uu_x)+\eta u_t-\frac{1}{3}P_x=0,\\
&P+\varkappa_1P_t+\eta P+3n\varkappa_1\eta_tP\\
&\quad=-(\beta_1+\beta_2)\eta_{tt}-(2\beta_2-\beta_1)\eta\eta_{tt}\\
&\qquad-\left(\frac{3n-1}{2}\beta_1+\frac{3}{2}\beta_2\right)\eta_t^2-\lambda\eta_t-3\eta+3\eta^2,
\end{aligned} \tag{2.11}$$

where the parameters are determined by formulae