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Noise suppress or express exponential growth for hybrid Hopfield neural networks

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ABSTRACT

In this Letter, we will show that noise can make the given hybrid Hopfield neural networks whose solution may grows exponentially become the new stochastic hybrid Hopfield neural networks whose solution will grows at most polynomially. On the other hand, we will also show that noise can make the given hybrid Hopfield neural networks whose solution grows at most polynomially become the new stochastic hybrid Hopfield neural networks whose solution will grows at exponentially. In other words, we will reveal that the noise can suppress or express exponential growth for hybrid Hopfield neural networks.

1. Introduction

It has been known that a given ordinary differential equation and its corresponding stochastic perturbed equation may have significant differences. The reason is that the noise can stabilize a given unstable system [1–9], as well as destabilize a given stable system [6,8]. A few years ago, another important fact that the environmental noise can suppress explosions in a finite time in population dynamics has been showed in [10], and this Letter made an important impact on the study of stochastic population systems. Recently, one more important feature are revealed that noise can suppress or express exponential growth of differential equation [11–13].

Hopfield neural networks (HNNs), which were introduced in 1980s [14,15], have drawn increasing interest over the past few decades owing to their broad applications in a variety of areas, such as signal processing, pattern recognition, associative memory, and combinational optimization. In the past few years, the dynamical behaviors of stochastic HNNs have emerged as a new subject of research mainly for two reasons: i) in real nervous systems, the synaptic transmission is a noisy process brought on by random fluctuations from the release of neurotransmitters and other probabilistic causes; ii) it has been realized that a neural network could be stabilized or destabilized by certain stochastic inputs [16,17]. In particular, the stability criteria for HNNs becomes an attractive research problem of prime importance [18–37]. In implementation or applications of neural networks, it is not uncommon for the parameters of neural networks (e.g. connection weights and biases) change abruptly due to unexpected failure or designed switching [38]. In such a case, neural networks can be represented by a switching model which can be regarded as a set of parametric configurations switching from one to another according to a given Markovian chain. So it is necessary to research the solution of stochastic switching Hopfield neural networks.

As a special switching systems, in this Letter, we addressed that to find the significant differences between the given hybrid HNNs and its corresponding perturbed by Itô stochastic hybrid HNNs. Like [12], we will consider the given hybrid HNNs under regime switching (coloured noise) whose solution may grows exponentially, and suppose that the hybrid HNNs are subject to environmental noise (white noise). We will reveal that regime switching and environmental noise work together to make the hybrid HNNs whose solution grows at most polynomially. That is to say noise will suppress the exponential growth for hybrid HNNs. On the other hand, every thing has two sides. We also shown that the given hybrid HNNs whose solution may grows polynomially can be perturbed by Itô stochastic hybrid HNNs whose solution grows exponentially. That is to say noise also can express exponential growth for hybrid HNNs.

2. Polynomial growth of stochastic hybrid Hopfield neural networks

Throughout this Letter, unless otherwise specified, R^n and $R^{n\times m}$ denote, respectively, the n-dimensional Euclidean space and the set of $n\times m$ real matrices. Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geqslant 0}, P)$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions (i.e. the filtration contains all p-null sets and is right

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continuous). $W(t)$ be a scalar Brownian motion defined on the probability space. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its operator norm is denoted by $\|A\| = \sup\{|Ax|: |x| = 1\}$, where $|\cdot|$ is the Euclidean norm. For a symmetric matrix A in $R^{n\times n}$, $\lambda_{min}(A)$ and $\lambda_{max}(A)$ mean the smallest and largest eigenvalue, respectively. If x, y are real numbers, then $x\vee y$ denotes the maximum of x and y, and $x\wedge y$ denotes the minimum of x and y. The shorthand $\mathrm{diag}\{M_1, M_2, \ldots, M_n\}$ denotes a block diagonal matrix with diagonal blocks being the matrices $M_1, M_2, \ldots, M_n$.

Let $r(t)$, $t \geqslant 0$, be a right-continuous Markov chain on the probability space taking values in a finite state-space $S = \{1, 2, \ldots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$P\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j, \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \geqslant 0$ is the transition rate from i to j if $i \neq j$, while $\gamma_{ii} = -\sum_{j\neq i}\gamma_{ij}$. We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $W(t)$. It is well known that almost every sample path of $r(t)$ is a right-continuous step function. As a standing hypothesis we assume in this Letter that the Markov chain is irreducible. This is equivalent to the condition that for any $i, j \in S$, one can find finite numbers $i_1, i_2, \ldots, i_k \in S$ such that $\gamma_{ii_1}\gamma_{i_1 i_2}\cdots\gamma_{i_k j} > 0$. Note that Γ always has an eigenvalue 0. The algebraic interpretation of irreducibility is $\mathrm{rank}(\Gamma) = N - 1$. Under this condition, the Markov chain has a unique stationary probability distribution $\pi = (\pi_1, \pi_2, \ldots, \pi_N) \in R^{1\times N}$ which can be determined by solving the following linear equation $\pi\Gamma = 0$. Subject to $\sum_{j=1}^N \pi_j = 1$ and $\pi_j > 0\ \forall j \in S$.

Consider an n-dimension stochastic Hopfield neural networks with Markovian switching

$$\begin{aligned} \mathrm{d}x(t) = {} & \big[-A\big(r(t)\big)x(t) + B\big(r(t)\big)f\big(x(t)\big)\big]\,\mathrm{d}t \\ & + g\big(x(t), r(t), t\big)\,\mathrm{d}W(t) \end{aligned} \tag{1}$$

on $t \geqslant 0$ with initial values $x(0) = x_0$ and $r(0) = i_0 \in S$, where $x(t) = (x_1(t), \ldots, x_n(t))^T$ is the state of the neuron, assume that origin is the equilibrium point, $A(i) = \mathrm{diag}\{a_1(i), \ldots, a_n(i)\}$ is the self-feedback connection weight matrix, $B(i) = (b_{kl}(i))_{n\times n}$ is connection weight matrix, $f(x(t)) = (f(x_1(t)), \ldots, f(x_n(t)))^T$ is a vector-valued activation function, and $g: R^n \times S \times R_+ \to R^{n\times n}$ which are satisfy the following assumptions.

Assumption 1. The activation function $f(\cdot)$ is bounded, and satisfy the following Lipschitz condition:

$$\big|f(u) - f(v)\big| \leqslant \big|G(u-v)\big| \quad \forall u, v \in R^n, \tag{2}$$

where $G \in R^{n\times n}$ is a known constant matrix.

Assumption 2. The coefficient g is locally Lipschitz continuous, that is, for each $k = 1, 2, \ldots,$ there is a positive number h_k such that $|g(x, i, t) - g(y, i, t)| \leqslant h_k|x - y|$ for all $i \in S$, $t \geqslant 0$ and those $x, y \in R^n$ with $|x| \vee |y| \leqslant k$. Assume also that g obey the linear growth condition, that is, there is a positive constant h such that $|g(x, i, t)|^2 \leqslant h(1 + |x|^2)$ for all $(x, i, t) \in R^n \times S \times R_+$.

It is known that under Assumptions 1 and 2, the stochastic hybrid HNNs have a unique global solution $x(t)$ on $t \in R_+$ and there is a positive constant H such that

$$\big|-A_i x + B_i f(x, i, t)\big|^2 \vee \big|g(x, i, t)\big|^2 \leqslant H\big(1 + |x|^2\big).$$

We also observe that the solution obeys

$$\limsup_{t\to\infty} \frac{1}{t}\log\big(\big|x(t)\big|\big) \leqslant \sqrt{H} + \frac{H}{2} \quad a.s.$$

(see e.g. [39, Theorem 3.17]). That is, the solution will grows at most exponentially with probability one. The following theorem shows that if the noise is sufficiently large, it will suppress this potentially exponential growth and make the solution grows at most polynomially.

Lemma 1. *(See [40].) Let $A = (a_{ij})_{n\times n} \in R^{n\times n}$. If $a_{ij} \leqslant 0$ for all $i \neq j$ and $\sum_{j=1}^n a_{ij} > 0$ for all $1 \leqslant i \leqslant n$, then $\det A > 0$.*

Lemma 2. *Let α, β, γ be three real numbers and define $\delta = \alpha\wedge\gamma$ if $\beta \geqslant 2(\alpha\wedge\gamma)$ or otherwise*

$$\delta = \min\left\{\alpha, \gamma, \frac{\alpha\gamma - 0.25\beta^2}{\alpha + \gamma - \beta}\right\}.$$

Then

$$\alpha + \beta u + \gamma u^2 \geqslant \delta(1+u)^2 \quad \forall u \geqslant 0. \tag{3}$$

Proof. If $\beta \geqslant 2(\alpha\wedge\gamma)$, then, for $u \geqslant 0$,

$$\alpha + \beta u + \gamma u^2 \geqslant \delta + 2\delta u + \delta u^2 = \delta(1+u)^2,$$

which is (3). Let us now consider the case when $\beta < 2(\alpha\wedge\gamma)$. Write, for $u \geqslant 0$,

$$\begin{aligned} & \alpha + \beta u + \gamma u^2 - \delta(1+u)^2 \\ & \quad = \alpha - \delta + (\beta - 2\gamma)u + (\gamma - \delta)u^2 \\ & \quad = (1, u)\begin{pmatrix} \alpha - \delta & 0.5\beta - \gamma \\ 0.5\beta - \gamma & \gamma - \delta \end{pmatrix}\begin{pmatrix} 1 \\ u \end{pmatrix}. \end{aligned}$$

By the definition of δ, we know $\alpha - \delta \geqslant 0$ and $\gamma - \delta \geqslant 0$. It is therefore clear that (3) will hold if

$$(\alpha - \delta)(\gamma - \delta) \geqslant (0.5\beta - \delta)^2,$$

namely

$$(\alpha + \gamma - \beta)\delta \leqslant \alpha\gamma - 0.25\beta^2.$$

But, given that $\beta < 2(\alpha\wedge\gamma)$, we must have $\beta < \alpha + \gamma$. Hence, the inequality above is equivalent to

$$\delta \leqslant \frac{\alpha\gamma - 0.25\beta^2}{\alpha + \gamma - \beta},$$

which is guaranteed by the definition of δ. In other words, we have shown that the assertion (3) holds as well when $\beta < 2(\alpha\wedge\gamma)$. □

Theorem 1. *Let Assumptions 1 and 2 hold. Assume that for each $i \in S$, there is a pair of positive constants ρ_i and α, such that*

$$\big|x^T g(x, i, t)\big|^2 \geqslant \rho_i|x|^4 - \alpha \tag{4}$$

for all $(x, t) \in R^n \times R_+$. Assume moreover that there is a constant $\theta \in (0, 1)$ such that

$$\begin{aligned} & \mathrm{diag}\big\{\theta\big[2(1-\theta)\rho_1 - h + 2\lambda_{min}(A_1) - \|B_1\|^2 - \|G\|^2\big], \ldots, \\ & \quad \theta\big[2(1-\theta)\rho_N - h + 2\lambda_{min}(A_N) - \|B_N\|^2 - \|G\|^2\big]\big\} - \Gamma \end{aligned} \tag{5}$$

is a nonsingular M-matrix. Then the solution of Eq. (1) obeys

$$\limsup_{t\to\infty} \frac{\log(|x(t)|)}{\log t} < \frac{1}{2\theta} \quad a.s. \tag{6}$$