Physics Letters A 374 (2010) 744–749

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Supersymmetric KdV–Sawada–Kotera–Ramani equation and its quasi-periodic wave solutions

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ARTICLE INFO

Article history:
Received 7 October 2009
Received in revised form 7 November 2009
Accepted 25 November 2009
Available online 2 December 2009
Communicated by R. Wu

PACS:
11.30.Pb
05.45.Yv
02.30.Gp
45.10.-b

Keywords:
Supersymmetric
KdV–Sawada–Kotera–Ramani equation
The Hirota's bilinear method
Super-Riemann theta function
Quasi-periodic wave solutions

ABSTRACT

In this Letter, we propose a supersymmetric KdV–Sawada–Kotera–Ramani equation. Based on a super-Riemann theta function, we devise a lucid and straightforward way for explicitly constructing a quasi-periodic wave solution of the supersymmetric KdV–Sawada–Kotera-Ramani equation. In addition, a one-soliton solution is obtained as a limiting case of the periodic wave solution under small amplitude. Indeed different from the purely bosonic case, the quasi-periodic wave observed shows that there is an "influencing band" among the waves under the presence of the Grassmann variable. The waves are symmetric about the band but collapse along with the band. Furthermore, the amplitudes of the waves increase as the waves move away from the band.

1. Introduction

The KdV–Sawada–Kotera–Ramani equation

$$u_t + a\big(3u^2 + u_{xx}\big)_x + b\big(15u^3 + 15uu_{xx} + u_{xxxx}\big)_x = 0 \tag{1.1}$$

was used to theoretically study the resonances of solitons in one-dimensional space by Hirota. It was found that two solitons near the resonant state exhibit some new phenomena [1]. The existence of conservation law for this equation was further proved by Konno [2]. In fact, Eq. (1.1) is a linear combination of the KdV equation and the Sawada–Kotera equation, which reduces to the KdV equation for $b = 0$ and the Sawada–Kotera equation for $a = 0$, respectively.

It is well known that a number of integrable equations can be generalized into the supersymmetric analogues [3–10], among them are the $\mathcal{N} = 1$ supersymmetric KdV equation

$$\Phi_t + 3\mathcal{D}_x^2(\Phi\mathcal{D}_x\Phi) + \mathcal{D}_x^6\Phi = 0, \tag{1.2}$$

and the $\mathcal{N} = 1$ supersymmetric Sawada–Kotera–Ramani equation

$$\Phi_t + \mathcal{D}_x^2\big[10(\mathcal{D}_x\Phi)\mathcal{D}_x^4\Phi + 5\big(\mathcal{D}_x^5\Phi\big)\Phi + 15(\mathcal{D}_x\Phi)^2\Phi\big] + \mathcal{D}_x^{10}\Phi = 0, \tag{1.3}$$

where the differential operator $\mathcal{D}_x = \partial_\theta + \theta\partial_x$ is the super-derivative, and $\Phi = \Phi(x, t, \theta)$ is fermionic superfield depending on usual independent variable x, t and Grassmann variable θ. Therefore, motivated by above discussion, it is natural for us to propose an $\mathcal{N} = 1$ supersymmetric KdV–Sawada–Kotera–Ramani equation

$$\Phi_t + a\mathcal{D}_x^2\big[3(\Phi\mathcal{D}_x\Phi) + \mathcal{D}_x^4\Phi\big] + b\mathcal{D}_x^2\big[10(\mathcal{D}_x\Phi)\mathcal{D}_x^4\Phi + 5\big(\mathcal{D}_x^5\Phi\big)\Phi + 15(\mathcal{D}_x\Phi)^2\Phi + \mathcal{D}_x^8\Phi\big] = 0, \tag{1.4}$$

The supersymmetric KdV equation (1.2) was introduced by Manin, Radul and Mathieu [4,5]. In recent years, much attention has been given to its bi-Hamiltonian structure, Painlevé property, infinite many symmetries, Darboux transformation, Bäcklund transformation, bilinear form and multi-soliton solutions [3–6]. Castea, Liu and Manas have found soliton solutions of Eq. (1.2) by using Hirota method and Darboux transformation, respectively [3,9]. The supersymmetric Sawada–Kotera–Ramani equation (3.3) was first proposed by Carstea [3]. The soliton solutions, Lax representation and infinite conserved quantities of this equation have been further obtained recently [11,12]. To the knowledge of the author, the quasi-periodic solutions of Eqs. (1.2)–(1.4), which can

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0375-9601/$ – see front matter
doi:10.1016/j.physleta.2009.11.071

be considered as a generalization of the soliton solutions, remain essentially open by either algebro-geometric or bilinear method.

The bilinear derivative method developed by Hirota is a powerful approach for constructing exact solution of nonlinear equations [13–19]. Based on the Riemann theta functions, Nakamura extended the Hirota's bilinear method to directly construct a kind of quasi-periodic solutions of nonlinear equation [20,21], where the periodic wave solutions of the KdV equation and the Boussinesq equation were obtained. Further development was made to investigate the discrete Toda lattice, $(2+1)$-dimensional Kadomtsev–Petviashvili equation, Bogoyavlenskii's breaking soliton equation and two other equations in $(2+1)$ dimensions possessing Hirota bilinear forms [22–27].

In this Letter, we aim at further extending the Hirota's bilinear method to construct quasi-periodic wave solutions of the supersymmetric KdV–Sawada–Kotera–Ramani equation (1.4). Based on a super-Riemann theta function, we devise a lucid and straightforward formula to construct quasi-periodic wave solutions for a class of supersymmetric equations. Once a nonlinear equation is written in bilinear form, then its quasi-periodic wave solutions can be obtained directly by using the formula. This formula has overcome repetitive recursion and computation which must be performed for each equation in previous works [20–27].

The organization of this Letter is as follows. In Section 2, we briefly introduce a super-Hirota bilinear operator and a super-Riemann theta function. Especially we devise a key formula for constructing periodic wave solutions. In Section 3, as application of our formula, we construct an explicit quasi-periodic wave solution to Eq. (1.4). We further propose an effective limiting procedure to analyze asymptotic behavior of the quasi-periodic wave solution. It is rigorously shown that the known one-soliton solution can be obtained as limiting cases of the quasi-periodic wave solution under a small amplitude.

2. Super-Hirota operator and Riemann theta functions

In order to apply the Hirota bilinear method for constructing multi-periodic wave solutions of Eq. (1.4), we consider the following variable transformation

$$\Phi = 2\mathcal{D}_x^3 \ln f(x,t,\theta). \tag{2.1}$$

Substituting (2.1) into (1.4), we then get the following bilinear form

$$S_x G(D_x, D_t) f\cdot f = S_x\big(D_t + aD_x^3 + bD_x^5 + c\big) f\cdot f = 0, \tag{2.2}$$

where c is an integration constant. The Hirota bilinear differential operators D_x and D_t are defined by

$$D_x^m D_t^n f(x,t,\theta)\cdot g(x,t,\theta) \\ = (\partial_x - \partial_{x'})^m(\partial_t - \partial_{t'})^n f(x,t,\theta) g\big(x',t',\theta\big)\big|_{x'=x,t'=t}.$$

The super-Hirota bilinear operator is defined as [3]

$$S_x^N f(x,t,\theta)\cdot g(x,t,\theta) \\ = \sum_{j=0}^{N} (-1)^{j|f|+\frac{1}{2}j(j+1)} \begin{bmatrix} N \\ j \end{bmatrix} \mathcal{D}_x^{N-j} f(x,t,\theta)\mathcal{D}_x^j g(x,t,\theta),$$

where the super-binomial coefficients are defined by

$$\begin{bmatrix} N \\ j \end{bmatrix} = \begin{cases} \begin{pmatrix} [N/2] \\ [j/2] \end{pmatrix}, & \text{if } (N,j) \neq (0,1) \bmod 2, \\ 0, & \text{otherwise}. \end{cases}$$

$[k]$ is the integer part of the real number k ($[k] \leqslant k \leqslant [k]+1$), and $|f|$ is the Grassmann parity of the function f defined by

$$|f| = \begin{cases} 1, & \text{if } f \text{ is odd}, \\ 0, & \text{if } f \text{ is even}. \end{cases}$$

Proposition 1. *The Hirota bilinear operators D_x, D_t and super-Hirota bilinear operator S_x have properties* [3]

$$S_x^{2N} f\cdot g = D_x^N f\cdot g, \\ D_x^m D_t^n e^{\xi_1}\cdot e^{\xi_2} = (\alpha_1-\alpha_2)^m(\omega_1-\omega_2)^n e^{\xi_1+\xi_2}, \\ S_x^{2N+1} e^{\xi_1}\cdot e^{\xi_2} = \big[\sigma_1-\sigma_2+\theta(\alpha_1-\alpha_2)\big](\alpha_1-\alpha_2)^N e^{\xi_1+\xi_2},$$

where phase variable $\xi_j = \alpha_j x + \omega_j t + \delta_j$, and α_j, ω_j, σ_j, δ_j are parameters, $j = 1, 2$. More generally, we have

$$F(D_x, D_t) e^{\xi_1}\cdot e^{\xi_2} = F(\alpha_1-\alpha_2, \omega_1-\omega_2) e^{\xi_1+\xi_2}, \tag{2.3}$$

where $F(D_x, D_t)$ is a polynomial about operators D_x and D_t.

In the following, we introduce a super-Riemann theta function and discuss its quasi-periodicity, which plays a central role in this Letter [28]

$$H\begin{bmatrix} \varepsilon \\ s \end{bmatrix}(\boldsymbol{\xi},\theta) = \vartheta\begin{bmatrix} \varepsilon \\ s \end{bmatrix}(\boldsymbol{\xi}, \tau+\theta\sigma). \tag{2.4}$$

The ordinary theta function appearing here is defined by

$$\vartheta\begin{bmatrix} \varepsilon \\ s \end{bmatrix}(\boldsymbol{\xi},\tau) = \sum_{n\in\mathbb{Z}^N} \exp\big\{2\pi i(\xi+\varepsilon)(n+s) - \pi\tau(n+s)^2\big\},$$

where the integer value $n\in\mathbb{Z}$, complex $s,\varepsilon\in\mathbb{C}$, and complex phase variables $\xi\in\mathbb{C}$. The $\tau>0$ is called the period matrix. For the simplicity, when $s=\varepsilon=0$, we denote

$$H(\boldsymbol{\xi},\theta) = H\begin{bmatrix} 0 \\ 0 \end{bmatrix}(\boldsymbol{\xi},\tau,\theta).$$

Definition 1. A function $g(t)$ on $\mathbb{C}$ is said to be quasi-periodic in t with fundamental periods $T_1,\ldots,T_k\in\mathbb{C}$ if $T_1,\ldots,T_k$ are linearly dependent over $\mathbb{Z}$ and there exist a function $G(y_1,\ldots,y_k)$ in $\mathbb{C}^k$, such that

$$G(y_1,\ldots,y_j+T_j,\ldots,y_k) = G(y_1,\ldots,y_j,\ldots,y_k), \\ \quad \text{for all } (y_1,\ldots,y_k)\in\mathbb{C}^k, \\ G(t,\ldots,t,\ldots,t) = g(t).$$

In particular, $g(t)$ becomes periodic with the period T if and only if $T_j = m_j T$.

Proposition 2. *The super-theta function $H(\xi,\theta)$ has the periodic properties* [28]

$$H(\xi+1,\theta) = H(\xi,\theta), \\ H(\xi+i\tau+i\theta\sigma,\theta) = e^{-\pi(\tau+\theta\sigma)-2\pi i\xi} H(\xi,\theta). \tag{2.5}$$

We regard 1 and $i\tau+i\theta\sigma$ as periods of the theta function $H(\xi,\theta)$ with multipliers 1 and $e^{-\pi(\tau+\theta\sigma)-2\pi i\xi}$, respectively. Here, 1 is actually period of the theta function $H(\xi,\theta)$, but $i\tau+i\theta\sigma$ is the period of the function $\mathcal{D}_x\partial_\xi \ln H(\xi,\theta)$.

Proposition 3. *The meromorphic functions $f(\xi)$ on $\mathbb{C}$ are as follows*:

$$f(\xi) = \alpha\mathcal{D}_x\partial_\xi \ln H(\xi,\theta), \quad \xi\in\mathbb{C}^N,$$

then it holds that

$$f(\xi+i\tau+i\theta\sigma) = f(\xi), \quad \xi\in\mathbb{C}. \tag{2.6}$$