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Bifurcation structure of successive torus doubling

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Abstract

The authors discuss the “embryology” of successive torus doubling via the bifurcation theory, and assert that the coupled map of a logistic map and a circle map has a structure capable of generating infinite number of torus doublings.

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1. Introduction

In the field of chaos research, one of the most important problems is to clarify how periodic attractors bifurcate to chaos. Torus doubling is one of the most interesting transitions from torus to chaos. Torus doubling was discovered in the early 1980s [1–6], and has been studied intensively with great interest [7–13]. Most researchers have believed that torus doubling occurs finite number of times, and that chaos arises from the oscillation of torus after finite number of torus doublings [3–6]. Such a belief differs from that of period doubling bifurcations. In particular, Kaneko proposed a coupled map of a logistic map that generates period doubling bifurcations and a linear circle map that generates tori, and asserted that the basic mechanism of torus doubling can be observed in this coupled map [3, 4]. According to detailed numerical analysis, Kaneko showed that torus doubling occurs only finite number of times in the coupled map. Such intensive investigation of the coupled map where torus doubling occurs finite number of times has been widely recognized.

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It is indeed an important nonlinear problem to clarify whether torus doubling always occurs finite number of times or whether it could occur infinite number of times in some cases. Yoshinaga (one of the authors of this Letter) and Kawakami analyzed the driven modified Bonhöffer van der Pol (BvP) equation (a three-dimensional extension of the BvP equation with periodic force), and the bifurcational mechanism of the novel transition is explained while considering a cascade of codimension-two bifurcations of periodic oscillations [14]. They also suggested that a cascade of torus doubling occurs at the edge regions where saddle–node bifurcation curves have a *treelike pattern* [14,15]. However, they have not yet clarified the detailed bifurcation structure in the vicinities of saddle–node bifurcation curves. On the other hand, the authors found successive torus doubling, and called it "swollen shape type bifurcation" [11]. This type of torus doubling is found to occur successively in the neighborhood of the parameters where the torus is resonant. Such a torus doubling can be observed in both numerical simulation and laboratory experiment [11], even though this torus doubling has a delicate structure. The authors consider that the swollen shape type bifurcation suggests that torus doubling occurs infinite number of times. In Kaneko's coupled map, torus doubling is analyzed with a linear circle map. However, since the swollen shape type bifurcation is closely associated with frequency locking, it is considered that this phenomenon is closely associated with Arnold's tongues. Therefore, Kaneko's coupled map is an inappropriate model to cover all aspects of successive torus doubling because the linear circle map does not have entrainments. The authors consider using a sin-circle map instead of the linear circle map in Kaneko's coupled map.

In this Letter, the proposed coupled map of a logistic map and a sin-circle map has the form

$$T:(x_n, y_n)^T \to (x_{n+1}, y_{n+1})^T,$$
$$\begin{cases} x_{n+1} = ax_n(1-x_n) + \varepsilon_1 f(x_n, y_n), \\ y_{n+1} = y_n + b\sin 2\pi y_n + c + \varepsilon_2 g(x_n, y_n) \pmod 1, \end{cases} \tag{1}$$

where

$$f(x_n, y_n) = \sin 2\pi y_n \quad \text{and} \quad g(x_n, y_n) = x_n.$$

The bifurcation structure of successive torus doubling is investigated in this mapping, where swollen shape type bifurcation is actually observed successively in numerical simulation. It is extremely difficult to draw the boundary for aperiodic oscillations such as torus or chaos theoretically in a bifurcation diagram in general. However, swollen shape type bifurcation occurs at the edge regions where a torus becomes resonant and the boundary is saddle-node bifurcation. The bifurcation points of saddle-node bifurcations can be obtained theoretically [16]. The authors suggested that the accumulation points of swollen shape type bifurcation are codimension-two bifurcation points where period doubling bifurcation and saddle-node bifurcation occur simultaneously [11]. The codimension-two bifurcation points can also be obtained theoretically [15]. Therefore, by tracing the codimension-two bifurcation points, it is highly expected that the bifurcation structure of successive torus doubling can be clarified. Actually, it has been observed that the codimension-two bifurcations occur successively in this mapping, and that the parameter tends to converge to Feigenbaum's universal constant. These indicate that the coupled map of a logistic map and a sin-circle map has a structure capable of generating infinite number of torus doublings.

2. Analysis of coupled map of logistic map and sin-circle map

In the following, we fix the parameters b, ε_1, and ε_2 at 0.1, 0.0007, and 0.0007, respectively. In the analysis of torus doubling in a coupled map of a logistic map and a circle map, the small values of the coupling parameters ε_1 and ε_2 can be naturally assumed [3,4]. The reason for this is as follows: for example, see Ref. [17]. In Ref. [17], an autonomous circuit that generates torus doubling bifurcations route to chaos is analyzed, and all the Lyapunov exponents of the Poincaré map are calculated. Since the Poincaré map is discrete, the attractor is a torus if the largest (the first) Lyapunov exponent λ_1 is zero. Pay attention to the behavior of the second-dimensional Lyapunov exponent $\lambda_1 + \lambda_2$ where λ_2 is the second Lyapunov exponent. Look at the behavior of the second-dimensional Lyapunov exponent after the end of torus doubling. From Fig. 8 in Ref. [17], the second-dimensional Lyapunov exponent almost equals λ_1. Therefore, λ_2 almost equals zero even if the attractor is chaotic. This means the