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Are ghost surfaces quadratic-flux-minimizing?

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ABSTRACT

Two candidates for "almost-invariant" toroidal surfaces passing through magnetic islands, namely quadratic-flux-minimizing (QFMin) surfaces and ghost surfaces, use families of periodic pseudo-orbits (i.e. paths for which the action is not exactly extremal). QFMin pseudo-orbits, which are coordinate-dependent, are field lines obtained from a modified magnetic field, and ghost-surface pseudo-orbits are obtained by displacing closed field lines in the direction of steepest descent of magnetic action, $\oint \mathbf{A}\cdot\mathbf{dl}$. A generalized Hamiltonian definition of ghost surfaces is given and specialized to the usual Lagrangian definition. A modified Hamilton's Principle is introduced that allows the use of Lagrangian integration for calculation of the QFMin pseudo-orbits. Numerical calculations show QFMin and Lagrangian ghost surfaces give very similar results for a chaotic magnetic field perturbed from an integrable case, and this is explained using a perturbative construction of an auxiliary poloidal angle for which QFMin and Lagrangian ghost surfaces are the same up to second order. While presented in the context of 3-dimensional magnetic field line systems, the concepts are applicable to defining almost-invariant tori in other $1\frac{1}{2}$ degree-of-freedom nonintegrable Lagrangian/Hamiltonian systems.

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1. Introduction

The understanding of nonintegrable Hamiltonian systems is greatly simplified if one can construct a coordinate framework based on a set of surfaces that are either invariant under the dynamics or, where this is impossible, surfaces that are *almost-invariant*. As invariant tori and cantori in nonintegrable systems can be approximated by sequences of periodic orbits, the theory of almost-invariant surfaces is built around periodic orbits, which constitute the remanent invariant sets surviving after integrability is destroyed by symmetry-breaking perturbations. We consider two classes of almost-invariant surfaces, *quadratic-flux-minimizing* (QFMin) surfaces [1] and *ghost surfaces* [2,3].

Almost-invariant tori are important in the theory of magnetic confinement of toroidal plasmas, in particular to the theory of transport in chaotic magnetic fields [4], and we set this Letter in the context of the nonintegrable magnetic fields, **B**, encountered in devices without a continuous symmetry. However, as magnetic field lines are orbits of a $1\frac{1}{2}$ degree-of-freedom Hamiltonian system, [5] the discussion is applicable, with appropriate translations of terminology, to any such system—e.g. in this Letter we use "magnetic field line" and "orbit" interchangeably.

In Section 2 we introduce our general, arbitrary background toroidal coordinate system s,θ,ζ, and an auxiliary poloidal angle $\Theta(s,\theta,\zeta)$ that allows us to define the quadratic flux in a form independent of the choice of θ. In Section 3 we introduce the magnetic action integral, its first and second variations and Hamilton's Principle, while in Section 4 we introduce QFMin and (generalized) ghost-surface pseudo-orbits as alternative strategies for continuously deforming the action-minimax orbit associated with an island chain into the corresponding action-minimizing orbit.

In Section 5 we present numerical results for field-line Hamiltonians of the form $\chi_0(\psi)+\epsilon\chi_1(\psi,\theta,\zeta)$, where the flux function ψ plays the role of a momentum canonically conjugate to θ, and ϵ parametrizes the strength of the perturbation away from the integrable case described by the action-angle Hamiltonian χ_0. Plots are presented comparing the uncorrected (i.e. with $\Theta=\theta$) QFMin and Lagrangian ghost curves of Ref. [2], superposed on field-line puncture plots in a Poincaré surface of section. Two cases with different strengths of perturbation are shown, both quite strongly chaotic and both showing that the differences between even uncorrected QFMin and ghost curves are very small (except for some higher-order surfaces, in the more strongly chaotic case). This suggests that the two, seemingly very different, approaches to defining almost-invariant tori may be unified by appropriate choice of Θ, and that this will differ from θ by an amount small in ϵ.

In Section 6 we introduce a modified form of Hamilton's Principle that gives QFMin pseudo-orbits as extremizers of a pseudoaction. Section 7 gives the canonical, Hamiltonian form of this

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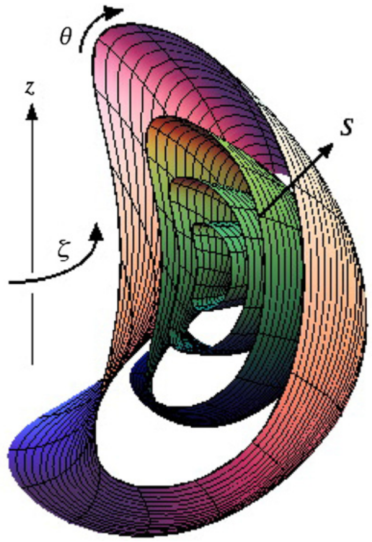

Fig. 1. A sketch of the general curvilinear toroidal coordinate system described in the text. (Color online.)

action principle, while Section 8 discusses the transformation to the Lagrangian form. In Section 9 we derive a consistency condition that Θ must satisfy for corrected QFMin surfaces to be Lagrangian ghost surfaces, finding in Section 10 an expression for a choice of the auxiliary angle Θ that satisfies this criterion up to first order in ϵ. The difference between uncorrected QFMin and ghost/corrected-QFMin pseudo-orbits is shown indeed to be very small, $O(\epsilon^2)$.

In Section 11 we sketch our finite-element variational method for numerical construction of QFMin surfaces using the new Hamilton's Principle introduced in Section 6, and in Section 12 we discuss the numerical construction of ghost surfaces via Galerkin projection onto the finite element basis.

Appendix A contains a derivation of the Euler–Lagrange equation for QFMin pseudo-orbits in the canonical representation, and Appendix B shows the relation between the generalized definition of ghost pseudo-orbit given in Section 4 and our more standard Lagrangian form [2], used in the numerical work and in Section 9.

2. Coordinates and fluxes

As depicted in Fig. 1, we assume a general, essentially arbitrary curvilinear toroidal coordinate system $s(\mathbf{r})$, $\theta(\mathbf{r})$, $\zeta(\mathbf{r})$ has been established, where $\mathbf{r}$ is a point in Euclidean 3-space and θ and ζ are respectively poloidal and toroidal angles labeling points on the toroidal isosurfaces of s, nested around the curve along which θ is singular (s increasing outward). We assume the nonorthogonal basis $\{\mathbf{e}^s, \mathbf{e}^\theta, \mathbf{e}^\zeta\} \equiv \{\nabla s, \nabla\theta, \nabla\zeta\}$ is right handed, as is its reciprocal basis $\{\mathbf{e}_s, \mathbf{e}_\theta, \mathbf{e}_\zeta\} \equiv \{\partial_s\mathbf{r}, \partial_\theta\mathbf{r}, \partial_\zeta\mathbf{r}\}$.

The directed infinitesimal area element on an arbitrary surface Γ is $d\mathbf{S} \equiv d\theta\, d\zeta\, \mathbf{n}/\mathbf{n}\cdot\nabla\theta \times \nabla\zeta$, where $\mathbf{n}$ is the unit normal at any point on Γ. Thus the net magnetic flux crossing an arbitrary torus Γ (which we assume to contain the θ-coordinate singularity curve) is

$$\varphi_1[\Gamma] \equiv \int_0^{2\pi}\int_0^{2\pi} d\theta\, d\zeta \frac{\mathbf{n}\cdot\mathbf{B}}{\mathbf{n}\cdot\nabla\theta \times \nabla\zeta}. \tag{1}$$

This integral is independent of choice of coordinates. In fact the absence of magnetic monopoles implies that φ_1 vanishes identically, so it is independent of the choice of Γ also, whether it be a magnetic surface (invariant torus of the field-line flow) or otherwise.

Thus, to measure the amount by which Γ departs from being a magnetic surface, we are led to define the positive definite *quadratic flux* [1], defined with the aid of a new generalized poloidal angle $\Theta(s, \theta, \zeta)$,

$$\varphi_2[\Gamma] \equiv \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi} d\theta\, d\zeta \frac{\mathbf{n}\cdot\mathbf{B}}{\mathbf{n}\cdot\nabla\theta \times \nabla\zeta}\frac{\mathbf{n}\cdot\boldsymbol{B}}{\mathbf{n}\cdot\nabla\Theta \times \nabla\zeta}. \tag{2}$$

The quadratic flux φ_2 is independent of the choice of base coordinates s, θ, ζ, but depends on the choice of Θ.

In the numerical work presented in this Letter, Θ has been chosen equal to the given angle θ. However in the formal development we distinguish it from θ so we can explore the consequences of making different choices, in particular whether it can be chosen so that QFMin tori coincide with ghost tori.

3. Magnetic action integral

The field-line action $\mathcal{S}$ [6] is a functional of a path $\mathcal{C}$ in Euclidean 3-space, points on which we take to be labeled by the toroidal angle ζ, which thus takes on the role played by time in a more conventional Hamiltonian system. In this Letter we confine our attention to paths that are closed loops, with θ increasing by $2\pi p$ when ζ increases by $2\pi q$ (p and $q > 0$ being mutually prime integers), so the average rate of increase of θ along the path is the rational fraction p/q, where the angular frequency $\iota\!\!\!-$ is called the *rotational transform*.

The magnetic action is defined by

$$\mathcal{S}[\mathcal{C}] \equiv \int_{\mathcal{C}} \mathbf{A}\cdot\mathbf{dl} \equiv \int_0^{2\pi q} \mathbf{A}\cdot\dot{\mathbf{r}}\, d\zeta, \tag{3}$$

where the single-valued function $\mathbf{A}(\mathbf{r})$ is a magnetic vector potential for the magnetic field, $\mathbf{B} = \nabla \times \mathbf{A}$, and $\mathbf{dl} \equiv \dot{\mathbf{r}}\, d\zeta$ is an infinitesimal line element tangential to $\mathcal{C}$. A superscript dot denotes the total derivative with respect to ζ, so that $\dot{\mathbf{r}}\cdot\nabla\zeta \equiv 1$. *Hamilton's Principle* is the statement that $\mathcal{S}$ is stationary, with respect to variations $\delta\mathbf{r}$ of $\mathcal{C}$, when $\mathcal{C}$ is a segment of a physical orbit (in our case a magnetic field line). If $\mathcal{C}$ is an open segment the variations are to be taken holding the endpoints fixed, but if (as we assume) $\mathcal{C}$ is a closed loop then the variations are unconstrained because the endpoint contributions cancel. Then, after integration by parts, we have the expansion for the total change in $\mathcal{S}$

$$\Delta\mathcal{S} = \int_0^{2\pi q}\left(\delta\mathbf{r}\cdot\frac{\delta\mathcal{S}}{\delta\mathbf{r}} + \frac{1}{2}\delta\mathbf{r}\cdot\frac{\delta^2\mathcal{S}}{\delta\mathbf{r}\,\delta\mathbf{r}}\cdot\delta\mathbf{r} + \cdots\right)d\zeta, \tag{4}$$

where the first functional derivative is given by

$$\frac{\delta\mathcal{S}}{\delta\mathbf{r}} \equiv \mathbf{e}^s\frac{\delta\mathcal{S}}{\delta s} + \mathbf{e}^\theta\frac{\delta\mathcal{S}}{\delta\theta} + \mathbf{e}^\zeta\frac{\delta\mathcal{S}}{\delta\zeta} = \dot{\mathbf{r}} \times \mathbf{B}. \tag{5}$$

Hamilton's Principle is now readily verified: The Euler–Lagrange equation $\delta\mathcal{S}/\delta\mathbf{r} = 0$ is satisfied if $\dot{\mathbf{r}} = \mathbf{B}/B^\zeta$, i.e. on a magnetic field line.

The symmetrized Hessian operator is

$$2\frac{\delta^2\mathcal{S}}{\delta\mathbf{r}\,\delta\mathbf{r}} = -\frac{d}{d\zeta}\mathbf{B}\times\mathbf{I} - \mathbf{I}\times\mathbf{B}\frac{d}{d\zeta} + \dot{\mathbf{r}}\times(\nabla\mathbf{B})^{\mathrm{T}} - (\nabla\mathbf{B})\times\dot{\mathbf{r}}, \tag{6}$$

where $\mathbf{I} = \mathbf{e}_s\mathbf{e}^s + \mathbf{e}_\theta\mathbf{e}^\theta + \mathbf{e}_\zeta\mathbf{e}^\zeta = \mathbf{e}^s\mathbf{e}_s + \mathbf{e}^\theta\mathbf{e}_\theta + \mathbf{e}^\zeta\mathbf{e}_\zeta$ is the identity dyadic and superscript $^{\mathrm{T}}$ denotes the transpose.

Note also that variations $\delta\mathbf{r} = \mathbf{r}(\zeta + \delta\zeta) - \mathbf{r}(\zeta) = \mathbf{r} + \dot{\mathbf{r}}\delta\zeta + \frac{1}{2}\ddot{\mathbf{r}}(\delta\zeta)^2 + \cdots$ that simply relabel the path can be verified to leave $\mathcal{S}$ invariant for arbitrary $\delta\zeta(\zeta)$, as expected. Thus, to find a unique