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Grazing-induced crises in hybrid dynamical systems

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ABSTRACT

In hybrid dynamical systems including both continuous and discrete components, an interplay between a continuous trajectory and a discontinuity boundary can trigger a sudden qualitative change in the system dynamics. Grazing phenomena, which occur when a continuous trajectory hits a boundary tangentially, are well known as a representative of such phenomena. We demonstrate that a grazing phenomenon of a chaotic attractor can result in its sudden disappearance and initiate chaotic transients. The mechanism of this grazing-induced crisis is revealed in an illustrative example. Furthermore, we derive a formula to obtain the critical exponent of the power law on the mean duration of chaotic transients.

1. Introduction

Interactions between continuous dynamics and discrete events can be found in many real-world systems such as impacting mechanics [1], switched electronic circuits [2,3], impulsive systems [4], and intermittent medical therapies [5]. Hybrid dynamical systems theory provides a mathematical framework for treating these systems including both continuous and discrete components [6–8]. Although the control theory for some classes of hybrid systems has been developed significantly since the 1960s [9], rich nonlinear and bifurcation phenomena in hybrid systems are still noteworthy topics of intensive investigation in nonlinear science [10]. In fact, hybrid systems can exhibit nonlinear phenomena such as grazing, border-collision, sliding, and chattering, all of which cannot be observed in ordinary smooth dynamical systems [11]. In this Letter, our attention is particularly focused on one of such discontinuity-induced phenomena, namely, grazing phenomena of chaotic attractors.

Grazing contacts between a continuous trajectory and a discontinuity boundary have been intensively investigated to understand a sudden qualitative change of system dynamics. We first consider the following simple class of hybrid systems [11], to review grazing phenomena and related terminologies:

$$\dot{\mathbf{x}} = F(\mathbf{x}) \quad \text{if } \mathbf{x} \in S^+, \tag{1}$$

where

$$S^+ = \left\{\mathbf{x} \in \mathcal{D} \subset \mathbb{R}^n \,\middle|\, H(\mathbf{x}) > 0\right\}. \tag{2}$$

The continuous state vector $\mathbf{x}$ of system (1) is defined in some domain $\mathcal{D} \in \mathbb{R}^n$. The system function F and the scalar function H are supposed to be smooth and well defined in the open neighborhood of S^+. We assume that some discrete state transition occurs at the surface

$$\Sigma = \left\{\mathbf{x} \in \mathcal{D} \,\middle|\, H(\mathbf{x}) = 0\right\}. \tag{3}$$

In other words, smoothness of an orbit in S^+ is lost on Σ due to non-smooth processes such as impact, jump, switch, and sliding. The smooth orbit generated by system (1) is denoted by $\phi(\mathbf{x}_0, t)$ with the initial condition $\mathbf{x}_0$ in S^+, which evolves until the orbit strikes the discontinuity boundary Σ.

Next we assume that there exists a *grazing point* $\mathbf{x}^*$ at which a trajectory grazes Σ as illustrated in Fig. 1, where $S^- = \{\mathbf{x} \in \mathcal{D} \subset \mathbb{R}^n\colon\ H(\mathbf{x}) < 0\}$. The grazing trajectory through $\mathbf{x}^*$ is indicated by (a). The conditions of the grazing point are given as follows [11]:

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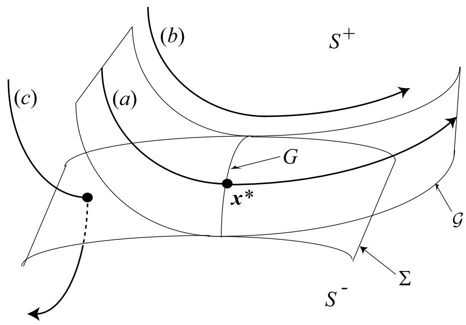

Fig. 1. Schematic illustration of the local neighborhood of the grazing point $\mathbf{x}^*$ in system (1). The grazing manifold $\mathcal{G}$ is the set of points on trajectories that have a grazing contact with Σ. The intersection of the grazing manifold and the discontinuity boundary Σ is denoted by $G = \mathcal{G} \cap \Sigma$. (a) A grazing trajectory through $\mathbf{x}^*$. (b) A trajectory which does not intersect with Σ. (c) A trajectory which transversally contacts with Σ and undergoes a discrete event.

$$H(\mathbf{x}^*) = 0, \tag{4}$$

$$\frac{\partial}{\partial t} H\big(\phi(\mathbf{x}^*, 0)\big) = 0, \tag{5}$$

$$\frac{\partial^2}{\partial t^2} H\big(\phi(\mathbf{x}^*, 0)\big) > 0. \tag{6}$$

The grazing point $\mathbf{x}^*$ is a point on Σ as described by Eq. (4). The condition (5) indicates that the grazing trajectory is just tangent to Σ at the grazing point. The open condition (6) is also necessary to guarantee that the grazing trajectory is locally a parabola that points downwards towards Σ.

A full local neighborhood of $\mathbf{x}^*$ can be divided into two regions, that is to say, the set of points on trajectories that do not intersect with Σ as indicated by (b) in Fig. 1 and the set of points on trajectories that do transversally intersect as indicated by (c) in Fig. 1. The destination of a trajectory after intersecting with Σ is highly influenced by the discrete event defined on Σ and a possible vector field defined in S^-. The boundary between the two regions is given by the *grazing manifold* $\mathcal{G}$ of points on trajectories that have a grazing contact. The intersection of $\mathcal{G}$ and Σ is the *grazing set*:

$$G = \mathcal{G} \cap \Sigma = \big\{\mathbf{x} \in \Sigma\colon (\partial H/\partial \mathbf{x})(dF/dt) = 0\big\}, \tag{7}$$

which corresponds to the set of grazing points.

A smooth variation of an initial condition of the flow ϕ can change transient dynamics qualitatively through a grazing phenomenon. This phenomenon is called *transient grazing* [12]. The critical initial condition $\mathbf{x}_0^*$ leading to transient grazing can be obtained by simultaneously solving Eqs. (4)–(5) and $\mathbf{x}^* = \phi(\mathbf{x}_0^*, \tau)$ with respect to unknown variables including $\mathbf{x}_0^*$ and the traveling time τ until the grazing contact, under the condition (6). Transient grazing has been mainly investigated with switching-time bifurcations so far [13].

A special case where a grazing trajectory is a periodic solution has received much more attention. We suppose that a limit cycle confined in S^+ exists near the boundary Σ at a certain value of a system parameter μ. As μ is smoothly varied, a grazing phenomenon of the limit cycle can take place at $\mu = \mu^*$. This is called a *grazing bifurcation* of a periodic solution. The grazing limit cycle can be specified by solving Eqs. (4)–(5) and the periodic condition $\mathbf{x}^* = \phi(\mathbf{x}^*, T)$ with respect to unknown variables including the grazing point $\mathbf{x}^*$, the period T of the limit cycle, and the critical parameter value μ^*, under the condition (6). Detailed analyses of grazing bifurcations of periodic solutions can be found in the studies of switched electronic circuits [14], power electronics [12], and intermittent medical therapies [15].

Grazing phenomena are closely related to border-collision bifurcations [16,17], which are also known as C-bifurcations [18]. Normal form maps can be derived from grazing orbits by considering an appropriate Poincaré section [19]. Despite abundant investigations on transient grazing and periodic grazing, few studies have dealt with grazing phenomena of chaotic solutions. The purpose of this Letter is to illustrate that a grazing phenomenon of a chaotic attractor can trigger a boundary crisis yielding chaotic transients. We uncover the mechanism of the grazing-induced crisis. Furthermore, the mean lifetime of chaotic transients caused by the grazing-induced crisis is characterized.

The rest of this Letter is organized as follows. In Section 2 we demonstrate a boundary crisis via which a chaotic attractor disappears in an illustrative example of hybrid systems. In Section 3 we reveal that the crisis is induced by a grazing phenomenon of the chaotic attractor. In Section 4 we characterize the duration of chaotic transients resulting from the grazing-induced crisis. The results are summarized in Section 5.

2. A boundary crisis in a hybrid dynamical system

In order to demonstrate a boundary crisis to be focused, we introduce a hybrid systems model representing prostate tumor growth under intermittent androgen suppression therapy [5,15,20]. The model assumes that a prostate tumor consists of androgen dependent (AD) cells and androgen independent (AI) cells. Androgen suppression is effective against AD cells but not for AI cells. Therefore, continuous androgen suppression often leads to a prostate cancer relapse due to an increase of AI cells under an androgen-depleted condition. An alternative treatment is intermittent androgen suppression, which aims to delay or hopefully prevent a cancer relapse by keeping a tumor susceptible to androgen suppression [5]. We suppose that androgen suppression is stopped when the serum prostate-specific antigen (PSA) value falls to a lower threshold level and re-initiated when the PSA value increases to an upper threshold level. On-treatment and off-treatment periods are alternately repeated in the intermittent therapy as long as possible. Since tumor dynamics is different between on-treatment and off-treatment periods, the time evolution of tumor growth can be described by a hybrid (switched) dynamical system as follows:

$$\dot{x} = \left(\alpha_x \frac{z + k_1 k_2}{z + k_2} - \beta_x \frac{z + k_3 k_4}{z + k_4} - m_{xy} \frac{z_0 - z}{z_0}\right) x,$$

$$\dot{y} = \left(m_{xy} \frac{z_0 - z}{z_0}\right) x + \big(\alpha_y (1 - ez) - \beta_y\big) y,$$

$$\dot{z} = -\big(z - z_0(1 - u)\big)/\tau_z, \tag{8}$$

$$u\colon \quad \begin{cases} 0 \to 1 & \text{if } v(x, y) = r_1 \text{ and } \dot{v}(x, y) > 0, \\ 1 \to 0 & \text{if } v(x, y) = r_0 \text{ and } \dot{v}(x, y) < 0, \end{cases} \tag{9}$$

where the positive continuous variables x, y, and z represent the population of AD cells, the population of AI cells, and the androgen concentration, respectively, and the discrete variable u represents on-treatment ($u = 1$) or off-treatment ($u = 0$). The parameters α_x, β_x, k_1, k_2, k_3, and k_4 are related to proliferation and apoptosis rates of AD cells. The parameters α_y, β_y, and e are related to proliferation and apoptosis rates of AI cells. The parameter m_{xy} is the rate of mutation of AD cells to AI cells. The parameters z_0 and τ_z are the maximum (normal) androgen level and the time constant of the androgen dynamics, respectively. Since the last equation of system (8) includes u, the continuous system is discontinuously switched by the rule (9) which depends on the monitored PSA value $v(x, y)$. For simplicity we assume $v(x, y) = x + y$. The parameters r_0 and r_1, satisfying $0 < r_0 < r_1$, represent the lower and upper threshold levels of the PSA value, respectively. For detailed backgrounds of the above model and its biological relevance, see Refs. [5,15].