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Coherent states of a particle in a magnetic field and the Stieltjes moment problem

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ABSTRACT

A solution to a version of the Stieltjes moment problem is presented. Using this solution, we construct a family of coherent states of a charged particle in a uniform magnetic field. We prove that these states form an overcomplete set that is normalized and resolves the unity. By the help of these coherent states we construct the Fock–Bergmann representation related to the particle quantization. This quantization procedure takes into account a circle topology of the classical motion.

1. Introduction

Constructing semiclassical or "classical-like" states in quantum mechanics is in general an open problem. For systems with quadratic Hamiltonians there exists a well-known procedure to construct the so-called coherent states (CS, or Glauber–Klauder–Sudarshan, or standard CS), which usually are accepted as quantum states that behave like their classical counterpart, see e.g. [1–4]. The CS are widely and fruitfully being utilized in different areas of theoretical physics. The standard CS turned out to be orbits of the Heisenberg–Weyl group. This observation allowed one to formulate by analogy some general definition of CS for any Lie group [5–8] as orbits of the group factorized with respect to a stationary subgroup. There exists a connection between the CS and the quantization of classical systems, in particular, systems with a curved phase space, see e.g. [9,10].

In [11] a modified approach to constructing semiclassical or coherent states (we call them also CS) was proposed. A technical realization of the approach recipes depends on each concrete case, in particular, a principal one is the problem of proving the resolution of the unity by the constructed CS. In the present article, we construct coherent states for a charged particle in a constant and uniform magnetic field closely by following the approach of [11], more exactly a squeezed version of these states. By solving a specific version of the Stieltjes moment problem we make explicit the resolution of the unity of the constructed CS and so become able to perform the Berezin–Klauder–Toeplitz or, more simply CS, quantization of the complex plane. A generalization of the obtained results to a model on a non-commutative plane will be the topic of a further work.

As far as physical applications are concerned, the resolution of the unity by CS is fundamental for the analysis, or decomposition, of states in the Hilbert space of the problem, or of operators acting on this space. In particular, it allows for a "classical" reading of quantum dynamical systems, in Schrödinger representation, through the time behavior of mean values of quantum observables in coherent states. Nice illustrations of this approach are provided by Perelomov in [5]. It was precisely this symbolic formulation that enabled Glauber and others to treat a quantized boson or fermion field like a classical field, particularly for computing correlation functions or other quantities of statistical physics, such as partition functions and derived quantities.

2. Coherent states of a particle in magnetic field

Consider a charged particle with charge e and mass μ placed in a uniform and constant magnetic field of magnitude B in the z-direction. The motion of the particle in a plane perpendicular to the magnetic field can be described by the quantum Hamiltonian ($c = \hbar = 1$)

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$$H = \frac{1}{2\mu}\left(P_1^2 + P_2^2\right), \quad P_i = p_i - eA_i,$$
$$A_i = -\frac{B}{2}\varepsilon_{ij}x^j, \quad i, j = 1, 2, \tag{1}$$

where x^i and p_i are canonical operators of coordinates and momenta of the particle and $\varepsilon_{ij}(\varepsilon_{12} = 1)$ is the Levi-Civita tensor. It is useful to introduce operators x_0^i, which are integrals of motion and correspond to the orbit center coordinates,

$$x_0^i = x^i + \frac{1}{\mu\omega}\varepsilon^{ij}P_j, \quad \omega = \frac{eB}{\mu},$$

and also the angular momentum operator of the relative motion J, which in the present case is just proportional to the Hamiltonian,

$$J = -\frac{1}{\omega}H = \frac{1}{2}\left(r^1P_2 - r^2P_1\right), \quad r^i = x^i - x_0^i.$$

Two independent Weyl–Heisenberg algebras underlie the symmetries and the integrability of the model. The first one concerns the operators $r_{0\pm} = x_0^1 \pm ix_0^2$ that obey $[r_{0-}, r_{0+}] = 2/\mu\omega$. The second one concerns the relative motion operators r^i, $r_\pm = r^1 \pm ir^2 = -[P_2 \mp iP_1]/\mu\omega$ with $[r_+, r_-] = 2/\mu\omega$. They allow one to construct a Fock space with orthonormal basis $\{|m, n\rangle\}$ obtained by repeated actions of the normalized raising operators:

$$\sqrt{\mu\omega/2}r_{0+}|m, n\rangle = \sqrt{m+1}|m+1, n\rangle,$$
$$\sqrt{\mu\omega/2}r_-|m, n\rangle = \sqrt{n+1}|m, n+1\rangle.$$

Like in [11], the CS $|z_0, \zeta\rangle$ are introduced as solutions of the eigenvalue problems

$$r_{0-}|z_0, \zeta\rangle = z_0|z_0, \zeta\rangle,$$
$$Z|z_0, \zeta\rangle = \zeta|z_0, \zeta\rangle, \quad z_0, \zeta \in \mathbb{C}, \tag{2}$$

with $Z = e^{-J+\frac{1}{2}}r_+$. The commutation relations $[J, r_\pm] = \pm r_\pm$ reproduce the appropriate algebra to study the circular motion, see [12]. These normalized CS are tensor product of the state $|\zeta\rangle$, that is an eigenvector of Z with the standard CS $|z_0\rangle$. They read in terms of the Fock basis,

$$|z_0, \zeta\rangle = \frac{1}{\sqrt{\mathcal{N}(z_0, \zeta)}} \times \sum_{m,n=0}^{\infty}\left(\frac{\mu\omega}{2}\right)^{\frac{m+n}{2}}\frac{z_0^m}{\sqrt{m!}}\frac{\zeta^n e^{-\frac{1}{2}(n+\frac{1}{2})^2}}{\sqrt{n!}}|m, n\rangle, \tag{3}$$

where $\mathcal{N}$ stands for a normalization factor.

One can easily see that the time evolution of the CS states is only reduced to the classical time evolution of the parameter ζ. Indeed, using the relations $[r_{0\pm}, H] = 0$ and $e^{itH}Ze^{-itH} = e^{-i\omega t}Z$, and applying the evolution operator $U(t) = \exp(-iHt)$ to the state $|z_0, \zeta\rangle$ we obtain

$$U(t)|z_0, \zeta\rangle = \left|z_0, \zeta(t)\right\rangle, \quad \zeta(t) = \zeta\exp(-i\omega t).$$

In order to play their role of "classical-between-quantal" bridge, the states $|z_0, \zeta\rangle$ have to resolve the identity in the above Fock–Hilbert space. This specific (over-)completeness can be proved by resolving the identity in the corresponding Fock–Bergmann representation. As was stated in [11], the task is equivalent to solving the following moment problem

$$\int_0^\infty t^n\varpi_q(t)\,dt = n!q^{\frac{n(n+1)}{2}}, \quad q = e^2, \tag{4}$$

for an unknown weight function $\varpi(t)$. Let us generalize the above problem and, consequently, obtain a squeezed version of the CS (2), by introducing the following displacement operator

$$Z_\lambda = \exp\left[\frac{\lambda}{2}\left(\frac{1}{2} - J\right)\right]r_+. \tag{5}$$

This operator coincides with Z from (2) for $\lambda = 2$, and with just r_+ for $\lambda = 0$ (or $q = 1$), i.e., the case where we have the tensor product of standard coherent states, called in this context the Malkin–Man'ko CS [13]. For an arbitrary λ the operator Z_λ controls the dispersion relations of the angular moment and of the position operators. In this case, the construction of the resolution of identity from the eigenstates of the above operator, and consequently the proof that they form an (over-)complete set, is equivalent to solving the moment problem of the form

$$\int_0^\infty t^n\varpi_q(t)\,dt = n!q^{\frac{n(n+1)}{2}}, \quad q \equiv e^\lambda, \ \lambda \geqslant 0,$$

for some unknown weight function $\varpi_q(t)$. Below, we find ϖ_q for an arbitrary $q \in [1, \infty)$ and deal with the corresponding CS as the eigenstates of the operator Z_λ. In addition, we discuss the extension of ϖ_q for $0 < q \leqslant 1$.

3. Solving Stieltjes moment problem

Let us consider the classical phase space $\mathbb{C}^2 = \{\mathbf{x} = (z, \zeta), z \in \mathbb{C}, \zeta \in \mathbb{C}\}$ provided with the measure:

$$\mu(d\mathbf{x}) = e^{-|z|^2}\frac{d^2z}{\pi}\varpi_q\left(|\zeta|^2\right)\frac{d^2\zeta}{\pi}, \tag{6}$$

where d^2z and $d^2\zeta$ are the respective Lebesgue measures on the complex planes. The positive weight function $0 \leqslant t \mapsto \varpi_q(t)$ solves the following Stieltjes moment problem:

$$\int_0^\infty t^n\varpi_q(t)\,dt = x_n! = n!q^{\frac{n(n+1)}{2}}, \quad q \geqslant 1, \tag{7}$$

where $x_n \overset{\text{def}}{=} nq^n$ and we have adopted the generalized factorial notation as $x_n! \overset{\text{def}}{=} x_nx_{n-1}\cdots x_1$, $x_0! = 1$.

In the Hilbert space

$$L^2\left(\mathbb{C}^2, \mu(d\mathbf{x})\right) = L^2\left(\mathbb{C}, e^{-|z|^2}\frac{d^2z}{\pi}\right) \otimes L^2\left(\mathbb{C}, \varpi_q\left(|\zeta|^2\right)\frac{d^2\zeta}{\pi}\right)$$

we select the orthonormal set of functions

$$\Phi_{m,n}(\mathbf{x}) \overset{\text{def}}{=} \frac{\bar{z}^m}{\sqrt{m!}}\frac{\bar{\zeta}^n}{\sqrt{x_n!}}, \tag{8}$$

which we put in one-to-one correspondence with the elements $|m, n\rangle$, $m, n \in \mathbb{N}$, of an orthonormal basis of a separable Hilbert space $\mathcal{H}$. The states (8) obey a finite sum property for any $\mathbf{x} \in \mathbb{C}^2$:

$$\sum_{m,n\in\mathbb{N}}\left|\Phi_{m,n}(\mathbf{x})\right|^2 = e^{|z|^2}\mathcal{E}_q\left(|\zeta|^2\right) < \infty, \tag{9}$$

where $\mathcal{E}_q(t)$ is the generalized "exponential" built from the sequence x_n:

$$\mathcal{E}_q(t) = \sum_{n=0}^\infty\frac{t^n}{x_n!} = \sum_{n=0}^\infty q^{-\frac{n(n+1)}{2}}\frac{t^n}{n!}. \tag{10}$$

It is clear that, due to the condition $q \geqslant 1$, the convergence radius of this power series is infinite. On the other hand it is zero if $q < 1$. In the sequel we use the notation

$$\mathcal{N}(\mathbf{x}) \overset{\text{def}}{=} e^{|z|^2}\mathcal{E}_q\left(|\zeta|^2\right).$$