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Prescribing a multistage analytical method to a prey–predator dynamical system

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ABSTRACT

This article discusses the effectiveness of a fresh analytical method in solving a prey–predator problem, which is described as a system of two nonlinear ordinary differential equations. The method of interest is the multistage variational iteration method (MVIM), which provides a slight modification of the classical variational iteration method (VIM). We shall compare solutions of the classical VIM along with MVIM and match them against the conventional numerical method, Runge–Kutta (RK4) (fourth-order).

1. Introduction

To mathematically describe the full dynamics of any biological system is a daunting task. Most biological systems are formed by nonlinear ordinary differential equations. This Letter studies a mathematical model of a certain prey–predator model, which describes the natural habitual existence of rabbits and foxes. These two animals are modeled to live together where the foxes prey on rabbits while the rabbits survives on clovers. Assuming there are infinite supply of clovers, the population (over time) of both animals depends on its initial population, food supply and of course the parameters which govern the population dynamics modeled. For an explicit exposition of the mathematical modeling of this system, we refer readers to [1]. The said problem was solved by Adomian decomposition method by Biazar and Montazeri [2] and Chowdhury et al. [3] and also the power series method (PSM) which can be found in [4]. Rafei et al. [5] and Yusufoğlu and Erbaş [6] have introduced the variational iteration method (VIM) in solving this dynamical system. The prey–predator system is demonstrated in (1) and (2) below

$$\frac{dx}{dt} = x(a - by), \tag{1}$$

$$\frac{dy}{dt} = -y(c - dx), \tag{2}$$

where $x(t)$ and $y(t)$ represent the populations of rabbits and foxes, respectively at time t and a, b, c, d are known coefficients.

Our work concentrates on a newly modified version of VIM, which is called the multistage variational method (MVIM) and we will present comparative solutions with VIM and the numerical method, fourth-order Runge–Kutta method (RK4). We chose the conventional RK4 as our benchmark for comparison purposes as it is widely accepted and used. The MVIM possesses a time-marching algorithm which speeds the convergence of the solutions rapidly. It has been applied to numerous problems such as in [7–9]. In the event of this study, we will also highlight the limitations of VIM and ADM, presented by [2,4–6]. These traditional analytic methods only provide valid solutions in a very short time frame for this particular prey–predator system.

2. Ideas on VIM and the multistage approach

Solving a system of nonlinear ordinary differential equations (ODEs) or even partial differential equations (PDEs) requires hard work. It is a tough job when we have to rely on numerical integration, perturbation techniques, some particular transformations, linearization or discretization in order to obtain their approximate solutions. Numerical integration methods are more flexible than any analytical approach, but it has its disadvantages. They react quite sensitively on the selection of time-step size to be depend-

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able [10]. On the other hand, analytical techniques do offer some options in acquiring solutions but they most likely require some sort of linearization techniques to be successful.

VIM has generated much buzz in the mathematical world. It was a theory spun by Prof. He [11–17]. This analytical technique overcomes some of the disadvantages of other methods by producing some accurate results plus its convenience (to use). Furthermore, VIM can provide a continuous representation of the approximated solution, which allows better information of the solution over the time interval. The Runge–Kutta method (RK4), on the other hand, provide solutions in discretized form, only at two ends of the time interval, thereby making it complicated in achieving a continuous representation. VIM is fast emerging as an essential tool used in researches, thanks to the contribution of numerous researchers, such as, Wazwaz [18], Momani and Odibat [19,20], Abdou and Soliman [21,22], Moghimi and Hejazi [23], Batiha et al. [24,25] and Inc [26].

VIM has a simple approach. We illustrate its basic concepts with a general differential equation

$$Lu + Nu = g(t), \tag{3}$$

where L is a linear operator, N a nonlinear operator, and $g(x)$ an inhomogeneous or forcing term. This will allow us to construct a correction functional as below

$$u_{i,n+1}(t) = u_{i,n}(t) + \int_{t_0}^{t} \lambda\big\{Lu_{i,n}(s) + N\tilde{u}_{i,n}(s) - g(s)\big\}\,\mathrm{d}s, \tag{4}$$

where $i = 1, 2, \ldots, m$, λ is a general Lagrange multiplier [27], which will be identified using the variational theory, n denotes the nth approximation and $\tilde{u}_{i,n}$ is considered as a restricted variation whereby $\delta\tilde{u}_{i,n} = 0$. The Lagrange multipliers can be easily and precisely obtained for linear problems. However, for nonlinear problems, it is not as trivial. The nonlinear terms are treated as restricted variations such that the Lagrange multiplier can be determined as a simpler form.

We will show that the approximated solutions for (4) in the numerical simulations of this Letter are not valid for large t. Rationally, it is easy to conceptualize that to ensure validity of the approximations for large t, we need to solve the system under arbitrary initial conditions and treat (4) as an algorithm for approximating the solutions of (1)–(2) in a sequence of intervals [25], whereby the solution from $[t_0, t)$ will be derived by subdividing this interval into $[t_0, t_1), [t_1, t_2), \ldots, [t_{j-1}, t)$ and applying the recursive formula of (5) on each subinterval.

$$u_{i,n+1}(t) = u_{i,n}(t) + \int_{t^*}^{t} \lambda\big\{Lu_{i,n}(s) + N\tilde{u}_{i,n}(s) - g(s)\big\}\,\mathrm{d}s. \tag{5}$$

Notice that this strategy gives a new construction of the correction functional (5) with a variable t^* as the lower limit of the integration instead of a fixed lower limit of t_0 in (4). The fixed limits is a norm used in the classical VIM which can be seen in [15–17]. This is the main difference between the newly modified version introduced by Batiha et al. [25] and the standard VIM. These authors have applied the method on a class of nonlinear system of ordinary differential equations with much success.

The initial approximation in each interval is taken from the solution in the previous interval,

$$u_{i,0}(t) = u_i(t^*) = c_i^* \tag{6}$$

where t^* is the left-end point of each subinterval and c_i^* is denoted as the initial approximations for $i = 1, 2, \ldots, m$.

By knowing the first initial conditions one would be able to solve (5) for all unknowns $u_{i,n}(t)$ $(i = 1, 2, \ldots, m;\ n = 0, 1, \ldots)$. In order to carry out the iteration in every subinterval of equal length Δt, $[t_0, t_1)$, $[t_1, t_2), [t_2, t_3), \ldots, [t_{j-1}, t_j = t)$, we need to know the values of the following:

$$u_{i,0}^*(t) = u_i(t^*), \quad i = 1, 2, \ldots, m. \tag{7}$$

These informations are typically not directly attainable, but through the initial value $t^* = t_0$, we could derive all the initial approximations. This is done by taking the previous initial approximation from the nth-iterate of the preceding subinterval given by (4), i.e.

$$u_{i,0}^*(t) \simeq u_{i,n}(t^*), \quad i = 1, 2, \ldots, m \text{ and } t^* \in (t_0, t_j). \tag{8}$$

The scheme mentioned above is called the multistage VIM or MVIM. It offers accurate solutions over a longer time frame (more stable) compared to the standard VIM [25]. This distinctive strategy grants the iterative algorithm a time-marching scheme which significantly drives forward the convergence of the solutions precisely with great rapidity. Intuitively, the value of t^* increases according to the designated time-step size in each iteration computation. Each amplified time step will produce a new approximation value for the desired iteration step and the process is continued until the targeted time frame is achieved. In other words, each new initial approximation is deliberately substituted into the subsequent calculation, making it a fast-forwarding kind of solver. This makes sense as the more partitions are used, the more accurate the approximations (errors are minimized) are but the downside is having to deal with longer computational time. The solutions should converge to accurate values as the number of iterative steps increases. Convergence to accurate solutions depends highly on the choice of initial approximations which then determine the number of iterates that give the most precise solution. Running the iterations on different time steps may provide us detailed information of its precision.

We shall construct the correction functional based on this simple modification to VIM on the system in (1) and (2) into such a form

$$x_{n+1}(t) = x_n(t) + \int_{t^*}^{t} \lambda_1(\tau)\left\{\frac{\mathrm{d}x_n}{\mathrm{d}\tau} - ax_n + b\widetilde{x_n y_n}\right\}\mathrm{d}\tau, \tag{9}$$

$$y_{n+1}(t) = y_n(t) + \int_{t^*}^{t} \lambda_2(\tau)\left\{\frac{\mathrm{d}y_n}{\mathrm{d}\tau} + cy_n - d\widetilde{x_n y_n}\right\}\mathrm{d}\tau, \tag{10}$$

where λ_1 and λ_2 are general Lagrange multipliers, n denotes the nth approximation, $\widetilde{x_n y_n}$ denotes restricted variations, i.e. $\delta\widetilde{x_n y_n} = 0$.

By taking variation with respect to the independent variables x_n and y_n, we will obtain the following:

$$\delta x_{n+1}(t) = \delta x_n(t) + \delta \int_{t^*}^{t} \lambda_1(\tau)\left\{\frac{\mathrm{d}x_n}{\mathrm{d}\tau} - ax_n + b\widetilde{x_n y_n}\right\}\mathrm{d}\tau, \tag{11}$$

$$\delta y_{n+1}(t) = \delta y_n(t) + \delta \int_{t^*}^{t} \lambda_2(\tau)\left\{\frac{\mathrm{d}y_n}{\mathrm{d}\tau} + cy_n - d\widetilde{x_n y_n}\right\}\mathrm{d}\tau. \tag{12}$$

Table 1
Four different cases of various parameters used in analysis.

Case	a	b	c	d	$x(0)$	$y(0)$
1	1	1	0.1	1	14	18
2	0.1	1	1	1	14	18
3	0.1	1	1	1	16	10
4	1	1	0.1	1	16	10