Physics Letters A 372 (2008) 7001–7006

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Explicit solutions of two nonlinear dispersive equations with variable coefficients

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ARTICLE INFO

Article history:
Received 6 August 2008
Accepted 17 October 2008
Available online 21 October 2008
Communicated by R. Wu

PACS:
52.35.Sb
52.25.Vy
05.45.Yv

Keywords:
Auxiliary equation method
Symbolic computation system
Explicit solutions
Camassa–Holm equation

ABSTRACT

A mathematical technique based on an auxiliary equation and the symbolic computation system Matlab is developed to construct the exact solutions for a generalized Camassa–Holm equation and a nonlinear dispersive equation with variable coefficients. It is shown that the variable coefficients of the derivative terms in the equations cause the qualitative change in the physical structures of the solutions.

1. Introduction

Camassa and Holm [1] derived a completely integrable wave equation

$$u_t - u_{xxt} + 3uu_x + 2ku_x = 2u_x u_{xx} + uu_{xxx} \tag{1}$$

by retaining two terms that are usually neglected in the small amplitude, shallow water limit. The function u is the fluid velocity in the x direction. The constant k is related to the critical shallow water wave speed. The term uu_{xxx} makes Eq. (1) nonlinear in its highest order derivatives and so it lies in the class of nonlinear dispersive wave models. Camassa and Holm [1] showed that for all k, Eq. (1) is integrable, and for $k = 0$, it has travelling wave solutions of the form $ce^{-|x-ct|}$, which are called peakons. As shown in [1,2], the Camassa–Holm equation (1) is bi-Hamiltonian, and hence admits an infinite hierarchy of symmetries and conservation laws.

Following Camassa and Holm's work, Eq. (1) and its various generalized forms have been studied extensively. Matsuno [3] employed a perturbation method to investigate the Camassa–Holm equation and obtained its cusp and loop soliton solutions. Wazwaz [4] acquired the solitary wave solutions for a modified form of the Camassa–Holm equation by using the tanh method and the sine–cosine method. Lai and Xu [5] investigated the compact and noncompact structures for two types of generalized Camassa–Holm equations and obtained their compactons, solitons, solitary patterns, periodic solutions and algebraic travelling wave solutions. Wang and Tang [6] employed some special phase orbits and find four new exact wave solutions for Eq. (1). Making use of the sub-ODE method and the generalized auxiliary equation method, Yomba [7,8] made some important discovery in finding the exact travelling wave solutions of the generalized Camassa–Holm equations and other types of partial differential equations. By using the integral factor techniques for solving differential equations, Zheng and Lai [9] obtain the exact travelling wave solutions for a generalized Camassa–Holm equation and a nonlinear dispersive equation with constant coefficients.

Motivated by the desire to extend the work made in [9], we write the following generalized Camassa–Holm equation with variable coefficients

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0375-9601/$ – see front matter
doi:10.1016/j.physleta.2008.10.031

$$u_t + 2k(t)u_x - u_{xxt} + a(t)uu_x = b(t)u_x u_{xx} + uu_{xxx}, \quad b \neq 0, -1, \tag{2}$$

and the nonlinear dispersive equation

$$u_t + 2k(t)u_x + a(t)uu_x - b_1(t)u_{xxx} = b(t)u_x u_{xx} - uu_{xxx}, \quad b \neq 0, 1, \tag{3}$$

where $a(t)$, $b(t)$, $b_1(t)$ and $k(t)$ are functions of t. When $a(t)$, $b(t)$, $b_1(t)$ and $k(t)$ become constants, various forms of exact travelling wave solutions for Eq. (2) were obtained in [6–8] and some exact travelling wave solutions for both Eqs. (2) and (3) were acquired in [9].

In this Letter, making use of the auxiliary differential technique, we obtain many exact solutions with wave variable $\xi = px + q(t)$ for Eqs. (2) and (3) where p is a constant and $q(t)$ depends on t. It might be said that the exact solutions obtained in this Letter are non-travelling wave solutions in the classical sense. When the variable coefficients appearing in Eqs. (2) and (3) become constants, the solutions include those presented in [9].

2. Brief description of the auxiliary differential equation method

To illustrate the auxiliary differential method, we consider the nonlinear partial differential equation

$$P(u, u_t, u_x, u_{xx}, u_{xt}, u_{tt}, \ldots) = 0 \tag{4}$$

by assuming its solution in the form

$$u(x,t) = f(t) + h(t)z(\xi), \quad \xi = p(t)x + q(t), \tag{5}$$

where $f(t)$, $h(t)$, $p(t)$ and $q(t)$ are all unknown functions of t. We let $z(\xi)$ satisfy the following auxiliary differential equation

$$\left(\frac{dz}{d\xi}\right)^2 = a_1 + a_2 z + a_3 z^2, \tag{6}$$

where a_i $(i = 1, 2, 3)$ are arbitrary real constants. The solutions of Eq. (6) are listed in Table 1 (see appendix of Jeffrey [10]).

Substituting Eqs. (5) and (6) into Eq. (4) and setting the coefficients of $z^j(\xi)$ $(j = 0, 1, 2, \ldots)$ and $x^s z^i(\xi)\sqrt{a_1 + a_2 z(\xi) + a_3 z^2(\xi)}$ $(s, i = 0, 1, 2, \ldots)$ to zero yield a set of algebraic equations for $f(t)$, $h(t)$, $p(t)$ and $q(t)$. Using the symbolic computation system Matlab to solve the algebraic equations, we obtain the explicit expressions of $f(t)$, $h(t)$, $p(t)$ and $q(t)$, from which we get the solution of Eq. (4).

3. Exact solutions for Eq. (2)

We assume that the solutions of Eq. (2) take the form

$$u(x,t) = f(t) + h(t)z(\xi), \quad \xi = p(t)x + q(t), \tag{7}$$

where $f(t)$, $h(t)$, $p(t)$ and $q(t)$ are functions of t and $z(\xi)$ satisfies Eq. (6).

Substituting Eqs. (6) and (7) into Eq. (2) and letting each coefficients of $x^s z^i(\xi)\sqrt{a_1 + a_2 z(\xi) + a_3 z^2(\xi)}$ $(s = 0, 1,\ i = 0, 1, 2)$ and $z^j(\xi)$ $(j = 0, 1)$ to be zero, we obtain the following equations with respect to unknown functions $f(t)$, $h(t)$, $p(t)$ and $q(t)$

$$f'(t) = 0, \qquad h'(t) = 0, \qquad p'(t) = 0, \tag{8}$$

$$a(t)h^2(t)p(t) - a_3 b(t)h^2(t)p^3(t) - a_3 h^2(t)p^3(t) = 0, \tag{9}$$

$$2k(t)h(t)p(t) - a_3 h(t)p^2(t)q'(t) + a(t)h(t)p(t)f(t) - \frac{a_2}{2}b(t)h^2(t)p^3(t) + h(t)q'(t) - a_3 h(t)p^3(t)f(t) = 0. \tag{10}$$

Solving the above algebraic equations with the help of Matlab, we acquire that $f(t)$, $h(t)$ and $p(t)$ have to be constants and

$$p = \pm\sqrt{\frac{a(t)}{a_3[b(t)+1]}}, \tag{11}$$

$$q(t) = \pm\int \frac{2fa_3 a(t)b(t) + 4a_3 k(t)[b(t)+1] - ha_2 b(t)a(t)}{2a_3[a(t) - b(t) - 1]}\sqrt{\frac{a(t)}{a_3[b(t)+1]}}\,dt, \tag{12}$$

where f, h, a_1, a_2 and a_3 are arbitrary constants with $a_3 \neq 0$, $h \neq 0$ and $a(t) - b(t) - 1 \neq 0$. From formula (11), we derive that factor $\sqrt{\frac{a(t)}{b(t)+1}}$ must maintain as a constant. We write $\frac{a(t)}{b(t)+1} = M =$ constant in this section.

It is noted that the solution $u(x,t)$ of Eq. (2) can be obtained from (11) and (12) together with the expressions of $z(\xi)$ in Table 1. From Table 1, we know that $z(\xi)$ has different forms determined by the values of constants a_1, a_2 and a_3. Therefore, it is necessary for us to discuss different situations relating to the solutions of Eq. (2)

$$u_{1.1}(x,t) = f - \frac{ha_2}{2a_3} + h\exp\left\{\varepsilon\sqrt{M}x + \varepsilon\sqrt{M}\int\left[\frac{2Mfa_3 b(t) - Ma_2 hb(t)}{2a_3(M-1)} + \frac{2k(t)}{(M-1)}\right]dt\right\},$$
$$a_3 \neq 0, \quad a_2^2 - 4a_1 a_3 = 0, \tag{13}$$

$$u_{1.2}(x,t) = f + \varepsilon h\sqrt{-\frac{a_1}{a_3}}\sin\left[\sqrt{-M}x + \sqrt{-M}\int\frac{Mfb(t) + 2k(t)}{M-1}\,dt\right],$$
$$a_1 \neq 0, \quad a_2 = 0, \quad M < 0, \tag{14}$$