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Orbital stability of peakons with nonvanishing boundary for CH and CH-γ equations ☆

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ABSTRACT

In this Letter, we consider the general expressions of peaked traveling wave solutions for CH and CH-γ equations. The orbital stability of these peakons are directly proved in the H^1 norm. Some previous results are extended.

1. Introduction

Camassa and Holm [1] derived a shallow water wave equation

$$u_t + 2ku_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{1}$$

which is called Camassa–Holm equation (CH equation). For $k = 0$, the above equation implies the following equation:

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}. \tag{2}$$

They showed that Eq. (2) has peaked solitary wave solutions

$$u_1(x,t) = ce^{-|x-ct|}, \tag{3}$$

which have discontinuous first derivative at the wave peak in contrast to the smoothness of most previously known specious of solitary waves and thus are called peakons. Eqs. (1) and (2) arise as models for shallow water waves [1,23]. The peakons capture a characteristic of the travelling waves of greatest height—exact travelling solutions of the governing equations for water waves with a peak at their crest [24–26]. Simper approximate shallow water models (like the classical Korteweg–de Vries equation) do not present travelling wave solutions with this feature. The peakons are to be understood as weak solutions in the sense of papers [27,28].

CH equation (1) has been studied in a great lot of papers (see, for instance, [2–17]), and many satisfactory results have been obtained. Other than peakons Eqs. (1), (2) model breaking waves by having smooth solutions which develop singularities in finite time in the form of breaking waves [1,8,29,30]. Here we review some results on stability of solitary waves. When $k = 0$, Constantin and Strauss [4] investigated the orbital stability of peakons (3) for Eq. (2), they proved the stability of peakons (3) in the H^1 norm by a direct method depending on the special structure of (2). Their relative results are presented as follows, in [4], $u(x,t) \in C([0,T); H^1(R))$ is called a solution to (2) if $u(x,t)$ is a solution of (2) in the sense of distribution and the following quantities

$$E(u) = \int_R \left(u^2 + u_x^2\right) dx \quad \text{and} \quad F(u) = \int_R \left(u^3 + uu_x^2\right) dx$$

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are conserved. Peakons (3) is noted as

$$u_1(x,t) = c\varphi(x-ct) = ce^{-|x-ct|},$$

and the main theorem in [4] reads

Theorem. *If $u \in C([0,T); H^1(R))$ is a solution to (2) with*

$$\|u(\cdot,0) - c\varphi\|_{H^1} < \left(\frac{\epsilon}{3c}\right)^4, \quad 0 < \epsilon < c,$$

then

$$\|u(\cdot,t) - c\varphi(\cdot - \xi(t))\|_{H^1} < \epsilon \quad \text{for } t \in (0,T),$$

where $\xi(t) \in R$ is any point where the function $u(\cdot,t)$ attains its maximum.

For $k \neq 0$, by using the abstract results of Grillakis [18], it was proved that all H^1 solitary wave solutions of (1) are orbital stable [5].

However, when $k \neq 0$ for (1), it has peakons not vanishing 0 as $|x| \to \infty$. For instance, in case of $k \neq 0$ and $c = k/2$, Liu and Qian [2] demonstrated that Eq. (1) has peakons of form

$$u_2(x,t) = (3k/2)e^{-|x-\frac{k}{2}t|} - k. \tag{4}$$

Further, in case of $k \neq 0$, Zhang [3] obtained general expression of peakons as follows:

$$u_3(x,t) = v(x,t) - k = (k+c)e^{-|x-ct|} - k. \tag{5}$$

Clearly, (3) and (4) are special cases of (5). Stability of such peakons as form (5) has not been solved yet.

In this Letter, firstly we consider the stability of general peakons (5) for Eq. (1). Using some transformation and the method in [4], we prove the orbital stability of peakons (5) in the H^1 norm. Secondly, by the relationship between CH equation and CH-γ equation (see Section 3) we show the orbital stability of the general peakons for CH-γ equation.

2. Stability of peakons for CH equation

In this section we prove the stability of peakons (5). The solutions $u_3(x,t)$ approach the constant $-k$ as $|x| \to \infty$. From the structure of peakons (5) we know that $u_3(x,t)$ may be regarded as the addition of functions $v(x,t)$ and a constant $-k$, where $v(x,t) = (k+c)e^{-|x-ct|} \to 0$ as $|x| \to \infty$, and $E(v)$ and $F(v)$ for $v(x,t)$ are conserved. So let

$$X_1 = \{u(x,t)\colon u(x,t) = \tilde{u}(x,t) - k\}$$

with $\tilde{u}(x,t) \in C([0,T); H^1(R))$ and $E(\tilde{u})$ and $F(\tilde{u})$ are conserved, k is the constant in Eq. (1).

Note $u_0 = u(x,0)$, $\tilde{u}_0 = \tilde{u}(x,0)$ and one of our results is given by

Theorem 1. *Suppose $\tilde{u}(x,t) \in C([0,T); H^1(R))$ and $u(x,t) = \tilde{u}(x,t) - k$ is a solution to CH equation (1). For any ϵ with $0 < \epsilon < k+c$, there exists a $\delta = \frac{\epsilon^4}{81(k+c)^3}$ such that if*

$$\|u_0 - u_3\|_{H^1} \leqslant \delta,$$

then

$$\|u(\cdot,t) - u_3(\cdot - \xi(t))\|_{H^1} < \epsilon \quad \text{for } t \in (0,T),$$

where $\xi(t) \in R$ is any point where the function $u(\cdot,t)$ attains its maximum.

Remark 1. In Theorem 1, $u(x,t)$ is called a solution to (1) if $u(x,t) \in X_1$ and $u(x,t)$ is a solution of (1) in the sense of distribution. For $k = 0$, $u(x,t)$ is namely the solution defined to (2) in [4], and Theorem 1 is restricted to the theorem in [4].

As (5) are solutions of CH equation (1), substituting (5) into (1) we get that $v(x-ct) = (k+c)e^{-|x-ct|}$ are solutions of equation

$$v_t - kv_x - v_{xxt} + 3vv_x + kv_{xxx} = 2v_xv_{xx} + vv_{xxx}. \tag{6}$$

Similarly, $\tilde{u}(x,t)$ is a solution of Eq. (6) on the assumption that $u(x,t) = \tilde{u}(x,t) - k$ is a solution to Eq. (1). Since

$$\|u_0 - u_3\|_{H^1} = \|\tilde{u}_0 - v\|_{H^1},$$
$$\|u(\cdot,t) - u_3(\cdot - \xi(t))\|_{H^1} = \|\tilde{u}(\cdot,t) - v(\cdot - \xi(t))\|_{H^1},$$

and $u(\cdot,t)$ attaining its maximum at $\xi(t)$ means $\tilde{u}(\cdot,t)$ attaining its maximum at $\xi(t)$, the question of stability of $u_3(x,t)$ for CH equation (1) can be reduced to the question of stability of $v(x-ct) = (k+c)e^{-|x-ct|}$ for Eq. (6) from the analysis above. In fact, Eq. (6) has the following conserved quantities:

$$E(v) = \int_r (v^2 + v_x^2)\,dx,$$

$$F_1(v) = \int_R (v^3 + vv_x^2 - kv^2 - kv_x^2)\,dx.$$

So $F(v) = \int_R (v^3 + vv_x^2)\,dx$ is also an invariant of Eq. (6).

Lemma 1. *For every $u \in X_1$ and $\xi \in R$*

$$\|u(\cdot,t) - u_3(\cdot - \xi)\|_{H^1}^2 = \|\tilde{u}(\cdot,t) - v(\cdot - \xi)\|_{H^1}^2$$
$$= E(\tilde{u}) - E(v) - 4(k+c)(\tilde{u}(\xi,t) - (k+c)).$$

Proof. By direct calculation we get $E(v) = 2(k+c)^2$ and

$$\begin{aligned}
&\|\tilde{u}(\cdot,t) - v(\cdot - \xi)\|_{H^1}^2 \\
&= E(\tilde{u}) + E(v) - 2\int_R \tilde{u}(x,t)v(x-\xi)\,dx - 2\int_R \tilde{u}_x(x,t)v_x(x-\xi)\,dx \\
&= E(\tilde{u}) + E(v) - 2\int_R \tilde{u}(x,t)v(x-\xi)\,dx - 2\int_{-\infty}^{\xi} \tilde{u}_x(x,t)v(x-\xi)\,dx \\
&\quad + 2\int_{\xi}^{\infty} \tilde{u}_x(x,t)v(x-\xi)\,dx \\
&= E(\tilde{u}) + E(v) - 4(k+c)\tilde{u}(\xi,t) \\
&= E(\tilde{u}) - E(v) - 4(k+c)(\tilde{u}(\xi,t) - (k+c)). \qquad \square
\end{aligned}$$

Lemma 2. *For $u = \tilde{u} - k$ in X_1, let $M = \max_{x \in R}\{\tilde{u}(x,t)\}$. If $\|u_0 - u_3\|_{H^1} \leqslant \delta$ for some $\delta = \delta_1(k+c)$ where $\delta_1 < \frac{1}{20}$, then*

$$|E(\tilde{u}) - E(v)| \leqslant \delta(2\sqrt{2}(k+c) + \delta) \quad \text{and}$$
$$|M - (k+c)| \leqslant 2\sqrt{(k+c)\delta}.$$

Proof.

(1) Owing to

$$E(v) = 2(k+c)^2, \qquad E(\tilde{u}(x,0)) = E(\tilde{u}(x,t)),$$

and

$$\|u_0 - u_3\|_{H^1} = \|\tilde{u}_0 - v\|_{H^1},$$