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Semigroup of positive maps for qudit states and entanglement in tomographic probability representation

V.I. Man'ko [a,*], G. Marmo [b], A. Simoni [b,*], F. Ventriglia [b]

[a] P.N.Lebedev Physical Institute, Leninskii Prospect 53, Moscow 119991, Russia
[b] Dipartimento Sc. Fisiche dell'Università Federico II e Sez. INFN di Napoli, Compl. Università di Monte S.Angelo, I-80126 Naples, Italy

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ABSTRACT

Stochastic and bistochastic matrices providing positive maps for spin states (for qudits) are shown to form semigroups with dense intersection with the Lie groups $IGL(n,\mathbb{R})$ and $GL(n,\mathbb{R})$ respectively. The density matrix of a qudit state is shown to be described by a spin tomogram determined by an orbit of the bistochastic semigroup acting on a simplex. A class of positive maps acting transitively on quantum states is introduced by relating stochastic and quantum stochastic maps in the tomographic setting. Finally, the entangled states of two qubits and Bell inequalities are given in the framework of the tomographic probability representation using the stochastic semigroup properties.

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1. Introduction

The description of a physical system admitting a probabilistic interpretation, be it classical or quantum, requires two collections of objects called states and observables, say $\mathcal{S}$ and $\mathcal{O}$ respectively, along with a pairing μ associating with any state ρ and observable A a Borel probability measure on the real line $\mathbb{R}$. If A is measured while the system is in a state ρ, $\mu_{A,\rho}$ represents the probability distribution for the observed values of A. Thus if $E\subseteq\mathbb{R}$ is a Borel set, $\mu_{A,\rho}(E)\in[0,1]$ is the probability that the measured value of A will be in the set E when the system is known to be in the state ρ. From a general point of view, above properties seem to be the minimal features that any physical system should possess.

This approach has been studied by several authors, for instance one can find a nice discussion by Mackey [1]. The set $\mathcal{S}$ describes the basic mathematical structure we are dealing with, while $\mu_{A,\rho}$ provides us with a physical interpretation. This probabilistic point of view is compatible with convex combinations on the space of states, indeed if ρ_1 and ρ_2 give rise to probability distributions, by setting $\mu_{A,\lambda\rho_1+(1-\lambda)\rho_2}=\lambda\mu_{A,\rho_1}+(1-\lambda)\mu_{A,\rho_2}$ we define a new probability distribution when $0\leqslant\lambda\leqslant 1$. Usually one requires some additional structure telling us how the system changes from time s to a later time t, i.e., requires the existence of a family of mappings $U_{t,s}:\mathcal{S}\to\mathcal{S}$ representing the dynamics and called evolution operator. The requirement that a state at a given time determines the state at a later time forces us to postulate the semigroup property

$$U_{t_2,t_1}=U_{t_2,s}\circ U_{s,t_1},\tag{1}$$

with $U_{t,t}$ the identity. Within this setting

$$\mu:\mathcal{S}\times\mathcal{O}\to\{\text{Borel probability measures on }\mathbb{R}\}.\tag{2}$$

A subset of observables is said to be a tomographic set τ if it allows to identify the state ρ (to "reconstruct" the state) when $\{\mu_{A,\rho}\}_{A\in\tau}$ is known. Very often τ is generated by acting with a group $\mathcal{G}$ on some fiducial observable A_0, i.e., it is the orbit of $\mathcal{G}$ in $\mathcal{O}$ through A_0. According to the group we use and the fiducial observable we start with, we deal with symplectic tomography, photon-number tomography and so on. While this approach is general enough to allow us to deal both with classical and quantum systems, here, to be more definite, we shall consider a quantum system with a finite number of levels.

States for quantum systems with a finite number of levels will be thought of as the spin states (or qudits) they can be described by density matrices which are hermitian nonnegative $(2j+1)\times(2j+1)$ matrices with unit trace. The linear maps of the spin states, positive maps, can be described by $(2j+1)^2\times(2j+1)^2$

* Corresponding authors.
E-mail addresses: manko@na.infn.it (V.I. Man'ko), marmo@na.infn.it (G. Marmo), simoni@na.infn.it (A. Simoni), ventriglia@na.infn.it (F. Ventriglia).

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matrices with special properties [2]. Recently it was shown [3–5] that qudit states can be described by probability distributions of random spin projection (called tomogram) depending on the direction of the quantization axis. In view of this the geometry of qudit states can be associated with the geometry of a simplex and the set of positive maps of qudit states can be associated with stochastic and bistochastic matrices moving points on the simplex. The aim of this work is to find the connection of spin tomograms with a unitary matrix containing eigenvectors of the density matrix of a qudit state and a point on the simplex which has the eigenvalues of the density matrix as its coordinates.

Another aim of the work is to define positive maps of qudit states through the transitive actions of both the unitary group on the eigenvectors of the density matrix and the stochastic matrix semigroup on the eigenvalues of the density matrix regarded as points of the simplex.

The qudit states of multipartite systems can be either separable or entangled. We formulate the properties of a qudit tomogram, which is the joint probability distribution of two spin projections on their own quantization axes, able to distinguish separable and entangled states. We consider the Bell inequalities [6,7] in the context of the properties of stochastic matrices constructed by using spin tomograms. The Cirelson [8] bound $2\sqrt{2}$ for the Bell-CHSH inequality of two qubits will be connected with some properties of a universal stochastic matrix obtained from the tomographic probability distribution describing maximally entangled two spin-1/2 states. The connection of positive maps with the semigroup of stochastic matrices provides the possibility to find a new relation of the maps with the Lie group of the general linear real transformations $GL(n,\mathbb{R})$ for bistochastic matrices and with the inhomogeneous group $IGL(n,\mathbb{R})$ for stochastic matrices.

This connection (which seems to have been unknown) provides a possibility to construct unitary representations of stochastic and bistochastic semigroups by reducing known infinite dimensional unitary irreducible representations of the Lie groups to the subsets of the Lie groups which are the semigroups under consideration.

The Letter is organized as follows: in Section 2 we review the spin tomography approach for one and two qudits. Examples of a qutrit and two qubit states in tomographic probability representation are studied in Section 3. The relation of stochastic and bistochastic semigroups with Lie groups is discussed in Section 4, mainly in the case a qutrit. In Section 5 a class of positive maps acting transitively on quantum states is introduced by relating stochastic and quantum stochastic maps in the tomographic setting. The relation of stochastic matrices with Bell inequality violation for entangled states of two qubits is discussed in Section 6. Some conclusions and perspectives are finally drawn in Section 7.

2. Spin tomograms and unitary group

As it was shown in [3,9,10] the qudit state described by a $(2j+1)\times(2j+1)$-matrix ρ can be also described by a tomographic probability distribution function, or tomogram, $\mathcal{W}(m,U)\geqslant 0$ where m is the spin projection: $m=-j,-j+1,\dots,j-1,j$; and U is a unitary $(2j+1)\times(2j+1)$-matrix. This matrix can be considered as a matrix of an irreducible representation of the rotation group depending on two Euler angles ϕ,θ determining the direction of quantization (or a point on the Bloch sphere S^2). The physical meaning of the tomogram $\mathcal{W}(m,U)$ is that, in the spin state with the given density matrix ρ, it gives the probability to obtain m as spin projection on the direction determined by the two angles ϕ,θ. It corresponds to choose $\{U^\dagger J_z U\}$ as tomographic set of isospectral observables, where $J_z=\sum_{m=-j}^{j} m|m\rangle\langle m|$ is one of the generators of the irreducible representation of the rotation group, so that $\mathcal{W}(m,U)$ is nothing but the value of the concentrated measure $\mu_{U^\dagger J_z U,\rho}$ at the spectral point m:

$$\mathcal{W}(m,U)=\mu_{U^\dagger J_z U,\rho}(m)=\operatorname{Tr}U^\dagger|m\rangle\langle m|U\rho=\langle m|U\rho U^\dagger|m\rangle. \tag{3}$$

The probability distribution is obviously nonnegative and normalized, i.e.,

$$\sum_{m=-j}^{j}\mathcal{W}(m,U)=1 \tag{4}$$

for any direction of the quantization axis. The spin tomogram can be also regarded as the diagonal matrix element of the rotated density matrix $U\rho U^\dagger$ in the natural basis $|m\rangle$.

The relation is invertible and knowing the tomogram $\mathcal{W}(m,U)$ for the matrices $U(\phi,\theta)$ of an irreducible representation of $SU(2)$ one obtains the density matrix ρ by means of a linear transform [9,10] which is the analog of the integral Radon transform but in the space of qudit states. Thus the quantum state of a qudit (a spin-j state) is known if the probability distribution $\mathcal{W}(m,U)$ of random spin projection as a function of the unitary matrix U is known. The tomogram $\mathcal{W}(m,U)$ can be used, consequently, in alternative to spinors (wave functions) or density matrices for describing spin states. The information on the spin state contained in the tomogram is redundant since it is sufficient to know the tomogram only for several directions determined by a set of angles $\{\phi_k,\theta_k\}$, whose number corresponds to the number of parameters determining the density matrix, equal to $(2j+1)^2-1$. But at the same time the dependence of the tomogram $\mathcal{W}(m,U)$ on the parameters of the unitary matrix U provides some advantage in considering the spins, $j=0,1/2,1,\dots$; and also the quantum states of several spins in an unified approach. For two qudits (spin j_1 and j_2) the tomogram of the quantum state with the $(2j_1+1)(2j_2+1)\times(2j_1+1)(2j_2+1)$ density matrix ρ is the normalized joint probability distribution

$$\mathcal{W}(m_1,m_2,U)=\langle m_1m_2|U\rho U^\dagger|m_1m_2\rangle \tag{5}$$

of two random spin projections $m_1=-j_1,-j_1+1,\dots,j_1-1,j_1$ and $m_2=-j_2,-j_2+1,\dots,j_2-1,j_2$ onto the corresponding directions determined by two pairs of Euler angles, ϕ_1,θ_1 and ϕ_2,θ_2. The information contained in the tomogram with a dependence on the matrix U of such a form is sufficient to reconstruct the density matrix ρ. But we define the tomogram by Eq. (3) to use the redundant information on the quantum state of bipartite systems in studying the entanglement properties of the system states. We remark that in Eq. (3) we could also use the full unitary group instead of $SU(2)$ and this we will do sometimes in the following.

The tomographic probability distribution of a qudit state $\mathcal{W}(m,U)$ can be considered as a column vector $\vec{\mathcal{W}}(U)$ with components

$$\begin{aligned}\mathcal{W}_1(U)&=\mathcal{W}(j,U),\mathcal{W}_2(U)\\&=\mathcal{W}(j-1,U),\dots,\mathcal{W}_{2j+1}(U)=\mathcal{W}(-j,U).\end{aligned} \tag{6}$$

Since all the components are nonnegative and the normalization condition (4) holds, from a geometrical point of view the components $\{\mathcal{W}_k\}$ of the tomographic probability vector determine the coordinates $\{x_k\}$ of points belonging to a simplex. For a qubit such a simplex is the segment $\{x_1+x_2=1;\ 0\leqslant x_1,x_2\leqslant 1\}$ in the plane x_1,x_2. For a generic qudit the simplex is a polyhedron in a $(2j+1)$-dimensional space determined by equations:

$$\sum_{k=1}^{2j+1}x_k=1,\quad 0\leqslant x_1,x_2,\dots,x_{2j+1}\leqslant 1. \tag{7}$$

Thus, the spin tomogram is a function of a unitary group element U with values in a simplex. The linear maps of probability vectors

$$\vec{\mathcal{W}}'(U)=M\vec{\mathcal{W}}(U) \tag{8}$$

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