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Application of a modified rational harmonic balance method for a class of strongly nonlinear oscillators

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ABSTRACT

An analytical approximate technique for conservative nonlinear oscillators is proposed. This method is a modification of the rational harmonic balance method in which analytical approximate solutions have rational form. This approach gives us the frequency of the motion as a function of the amplitude of oscillation. We find that this method works very well for the whole range of parameters, and excellent agreement of the approximate frequencies with the exact one has been demonstrated and discussed. The most significant features of this method are its simplicity and its excellent accuracy for the whole range of oscillation amplitude values and the results reveal that this technique is very effective and convenient for solving conservative truly nonlinear oscillatory systems with complex nonlinearities.

1. Introduction

Nonlinear oscillation in physics and applied mathematics has been a topic to intensive research for many years. There are a large variety of approximate methods commonly used for solving nonlinear oscillatory systems including perturbations [1–4], parameter-expansion [5,6], variational [7], variational iteration [8], homotopy perturbation [9–12], harmonic balance [1,13–17] methods, etc. Surveys of the literature with numerous references and useful bibliography and a review of these methods can be found in detail in [2] and [18]. In this Letter we apply a modified generalized, rational harmonic balance method to obtain analytic approximate solutions for a nonlinear oscillator for which the restoring force has a rational expression of the displacement. This method can be applied to nonlinear oscillatory systems where the nonlinear terms are not small and no perturbation parameter is required. In this method the approximate solution obtained approximates all of the harmonics in the exact solution [19], whereas the usual harmonic balance techniques provide an approximation to only the lowest harmonic components. In an attempt to provide better solution methodology a modification in this technique is proposed.

2. Formulation and solution method

To apply the modify rational harmonic balance method to we consider the following nonlinear oscillator

$$\frac{d^2x}{dt^2} + \frac{x}{1+x^2} = 0, \tag{1}$$

with initial conditions

$$x(0) = A \quad \text{and} \quad \frac{dx}{dt}(0) = 0. \tag{2}$$

Eq. (1) is an example of a conservative nonlinear oscillatory system in which the restoring force has a rational form

$$\frac{d^2x}{dt^2} = F(x), \quad F(x) = -\frac{x}{1+x^2}. \tag{3}$$

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All the motions corresponding to Eq. (1) are periodic [18] and the system will oscillates between symmetric bounds $[-A, A]$, and the frequency and corresponding periodic solution of the oscillator are dependent on the amplitude A. Note that for x, respectively, small and large, Eq. (1) becomes

$$x \text{ small} \quad \frac{d^2x}{dt^2} + x \approx 0, \qquad x \text{ large} \quad \frac{d^2x}{dt^2} + \frac{1}{x} \approx 0. \tag{4}$$

Thus, for x small, the equation of motion approximates that of a linear harmonic oscillator and its angular frequency is $\omega \approx 1$ for small A. However, for large x, the equation of motion is that of a truly nonlinear oscillator in which the restoring force is inversely proportional to the dependent variable and $\omega \approx \sqrt{2\pi}/2A = 1.25331/A$ [20–22], which tends to zero when A decreases. Consequently the angular frequency ω increases from 1 to $1.25331/A$ as the initial value of $x(0) = A$ increases. The main purpose of this Letter is to construct an analytical approximation to the solution of Eq. (1) using a modified rational harmonic balance method (RHBM) introduced by Beléndez et al. in Ref. [22] and which has been applied for truly conservative nonlinear oscillators with good results.

To solve Eq. (1) by the modified RHBM, a new independent variable $\tau = \omega t$ is introduced. Then Eqs. (1) and (2) can be rewritten as

$$\omega^2 \frac{d^2x(\tau)}{d\tau^2} + \frac{x(\tau)}{1 + x^2(\tau)} = 0, \tag{5}$$

$$x(0) = A, \qquad \frac{dx}{d\tau}(0) = 0. \tag{6}$$

The new independent variable is chosen in such a way that the solution of Eq. (5) is a periodic function of τ of period 2π. The corresponding frequency of the nonlinear oscillator is ω and it is a function of the amplitude A.

Following the lowest order harmonic balance approximation, we set $x_1(\tau) = A\cos\tau$ which satisfies the initial conditions in Eq. (6). Substituting $x_1(\tau)$ into Eq. (5) we obtain

$$-\omega^2 A\cos\tau + \frac{A\cos\tau}{1 + A^2\cos^2\tau} = 0. \tag{7}$$

We can do the following Fourier series expansion

$$\frac{A\cos\tau}{1 + A^2\cos^2\tau} = \sum_{n=0}^{\infty} a_{2n+1}\cos\big[(2n+1)\tau\big], \tag{8}$$

where

$$a_{2n+1} = \frac{4}{\pi}\int_0^{\pi/2} \frac{A\cos\tau}{1 + A^2\cos^2\tau}\cos\big[(2n+1)\tau\big]\,d\tau. \tag{9}$$

Substituting Eq. (8) into Eq. (7) gives

$$\big(-\omega^2 A + a_1\big)\cos\tau + \text{HOH} = 0, \tag{10}$$

where HOH stands for higher-order harmonics. Setting the coefficient of $\cos\tau$ to zero allows the determination of the first approximation to the frequency in terms of A

$$\omega_1(A) = \sqrt{\frac{a_1}{A}} = \sqrt{\frac{2}{A^2} - \frac{2}{A^2\sqrt{1 + A^2}}}. \tag{11}$$

In order to determine an improved approximation we use a rational form given by the following expression [1,19]

$$x_2(\tau) = \frac{A_1\cos\tau}{1 + B_2\cos 2\tau}. \tag{12}$$

In this equation A_1, B_2 and ω are to be determined as functions of the initial conditions expressed in Eq. (6) and $|B_2| < 1$. From Eq. (6) we obtain $A_1 = (1 + B_2)A$ and Eq. (12) can be rewritten as follows

$$x_2(\tau) = \frac{(1 + B_2)A\cos\tau}{1 + B_2\cos 2\tau}. \tag{13}$$

Substituting Eq. (13) into Eq. (5) leads to

$$-\omega^2\frac{A(1 + B_2)\cos\tau}{1 + B_2\cos 2\tau} + \omega^2\frac{4AB_2(1 + B_2)\cos 3\tau}{(1 + B_2\cos 2\tau)^2} + \omega^2\frac{8AB_2^2(1 + B_2)\cos\tau\sin^2 2\tau}{(1 + B_2\cos 2\tau)^3} + \frac{A(1 + B_2)(1 + B_2\cos 2\tau)\cos\tau}{(1 + B_2\cos 2\tau)^2 + A^2(1 + B_2)^2\cos^2\tau} = 0. \tag{14}$$

Eq. (14) can be written as follows

$$F(A, B_2, \omega, \tau) = 0. \tag{15}$$

As $|B_2| < 1$ we can do the following series expansion

$$F(A, B_2, \omega, \tau) = \sum_{n=0}^{\infty} f_n(A, \omega, \tau)B_2^n, \tag{16}$$