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# Multistability in bidirectional associative memory neural networks ☆

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**Abstract**

In this Letter, the multistability issue is studied for Bidirectional Associative Memory (BAM) neural networks. Based on the existence and stability analysis of the neural networks with or without delay, it is found that the $2n$-dimensional networks can have $3^n$ equilibria and $2^n$ equilibria of them are locally exponentially stable, where each layer of the BAM network has $n$ neurons. Furthermore, the results has been extended to $(n+m)$-dimensional BAM neural networks, where there are $n$ and $m$ neurons on the two layers respectively. Finally, two numerical examples are presented to illustrate the validity of our results.


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*Keywords:* Multistability; BAM neural networks; Delay; Equilibrium

## 1. Introduction

In recent years, neural networks have attracted more and more attention of researchers due to their great perspectives of application. Ranging from classifications, associative memory, image processing, and pattern recognition to parallel computation and its ability to solve optimization problems, neural networks work as an intelligent tool in different situations. Neural networks have complex dynamical behaviors, such as stability [1–5], periodic bifurcation and chaos [6–9], which have been extensively investigated. The theory on the dynamics of the networks have been developed according to the purposes of applications.

In the applications of neural networks for associative memory storage or pattern recognition, the coexistence of multiple equilibria is a necessary feature [10–13]. The notion of "multistability" of a neural network is used to describe coexistence of multiple stable patterns. In [14], the multistability of the delayed neural networks was discussed:

$$\dot{x}_i(t) = -\mu_i x_i(t) + \sum_{j=1}^{n} \alpha_{ij} g_j\big(x_j(t-\tau_{ij})\big) + J_i, \quad i = 1, 2, \ldots, n. \tag{1}$$

It is found that an $n$-neuron cellular neural networks can have up to $2^n$ locally stable equilibria. Ref. [15] studied a general delayed neural networks:

$$\dot{x}_i(t) = -\mu_i x_i(t) + \sum_{j=1}^{n} \alpha_{ij} g_j\big(x_j(t)\big) + \sum_{j=1}^{n} \beta_{ij} g_j\big(x_j(t-\tau_{ij})\big) + J_i, \quad i = 1, 2, \ldots, n. \tag{2}$$

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In Refs. [16–18], the multistability of cellular neural networks (CNNs) and delayed cellular neural networks (DCNNs) was investigated. Furthermore, in [15,17], the authors studied multiperiodicity and exponential attractivity of neural networks evoked by periodic external inputs. In [25], the multistability and multiperiodicity are discussed for a class of delayed Cohen–Grossberg neural networks. While many multistability criterions depend deeply on the self-connection weights $\alpha_{ii}$, $\beta_{ii}$ in the neural networks. For instance in [14–16], the related criterions always require $\alpha_{ii}$ or $\alpha_{ii} + \beta_{ii}$ to be positive. However, if $\alpha_{ii} = 0$, $\beta_{ii} = 0$, the criterions mentioned above are not applicable for checking the multistability of the neural networks.

Bidirectional Associative Memory (BAM) network, introduced by Kosko in [19–21], is a typical neural network model, in which the self-connections of all neurons are zero. It has been successfully applied to pattern recognition and associative memory. As an extension of the unidirectional autoassociator of Hopfield neural networks, BAM neural network is formed by neurons arranged in two layers. The neurons in one layer are fully interconnected to the neurons in the other layer, while there are no interconnections among neurons in the same layer. In Refs. [6,8,22,23], the authors discussed the problem of stability and periodic for BAM networks with or without axonal signal transmission delays. However, to the best of our knowledge, few papers (if any) are concerned with the multistability of BAM neural networks.

Motivated by the above discussions, we shall study the multistability of BAM neural networks in this letter. In Sections 2 and 3, the $2n$-dimensional networks are considered with $n$-neurons on each layer of the BAM networks. The condition of the existence of multiple equilibria is obtained, in Section 2. In Section 3, the stability of the equilibria is investigated with delay or without delay. In Section 4, the neural network model is extended to a more general form, in which there can be different number of neurons on the two layers. Both the global stability and local metastability conclusions are obtained. In Section 5, two illustrative examples are provided with simulation results. Finally, conclusions are given in Section 6.

## 2. Existence of multiple equilibria

In this section and Section 3, we consider the BAM neural networks without delay or with delay, respectively as follows:

$$\begin{cases} \dot{x}_i(t) = -a_i x_i(t) + \sum_{1\leqslant j\leqslant n} b_{ij} g(y_j(t)) + I_i, \\ \dot{y}_i(t) = -c_i y_i(t) + \sum_{1\leqslant j\leqslant n} d_{ij} g(x_j(t)) + J_i, \end{cases} \quad i = 1, 2, \ldots, n \tag{3}$$

$$\begin{cases} \dot{x}_i(t) = -a_i x_i(t) + \sum_{1\leqslant j\leqslant n} b_{ij} g(y_j(t-\tau_{ij})) + I_i, \\ \dot{y}_i(t) = -c_i y_i(t) + \sum_{1\leqslant j\leqslant n} d_{ij} g(x_j(t-\sigma_{ij})) + J_i, \end{cases} \quad i = 1, 2, \ldots, n \tag{4}$$

where $a_i > 0$, $c_i > 0$, $x_i$, $y_j$ are the activations of the $i$th neurons and $j$th neurons in the two layers, respectively. $b_{ij}$, $d_{ij}$ are the connection weights through the neurons in two layers, and $I_i$ and $J_i$ denote the external inputs. $\tau_{ij} > 0$, $\sigma_{ij} > 0$ correspond to finite speed of axonal signal transmission. Denote $\tau := \max_{1<i,j<n}\{\tau_{ij}, \sigma_{ij}\}$, where $\tau > 0$. The activation function $g(s) = \tanh(s)$, which holds the sigmoidal configuration and is nondecreasing with saturation. As a functional differential equations described by system (4), the initial condition is

$$\begin{cases} x_i(\theta) = \phi_i(\theta), \\ y_i(\theta) = \psi_i(\theta), \end{cases} \quad \theta \in [-\tau, 0],$$

where $\phi_i, \psi_i \in \mathcal{C}([-\tau, 0], \mathbb{R})$.

The stationary equations of systems (3) and (4) are identical as follows,

$$\begin{cases} -a_i x_i + \sum_{1\leqslant j\leqslant n} b_{ij} g(y_j) + I_i = 0, \\ -c_i y_i + \sum_{1\leqslant j\leqslant n} d_{ij} g(x_j) + J_i = 0, \end{cases} \quad i = 1, 2, \ldots, n. \tag{5}$$

Firstly, consider the BAM neural network with a single couple of neurons,

$$\begin{cases} \dot{x}(t) = -a_i x(t) + b_{ii} g(y(t)) + I_i, \\ \dot{y}(t) = -c_i y(t) + d_{ii} g(x(t)) + J_i. \end{cases} \tag{6}$$

Hence, the stationary equations can be rewritten as

$$\begin{cases} x = \frac{b_{ii}}{a_i} g(y) + \frac{I_i}{a_i} := G_i(y), \\ y = \frac{d_{ii}}{c_i} g(x) + \frac{J_i}{c_i} := H_i(x). \end{cases} \tag{7}$$

As is shown in Fig. 1, the equilibria of Eq. (6) are the crossing points of the curves $x = G_i(y)$ and $y = H_i(x)$.

Here, we propose the first condition:

$$(\mathrm{H}_1): \quad \frac{b_{ii} d_{ii}}{a_i c_i} > 1, \quad i = 1, 2, \ldots, n.$$

It's worth noting that, as $a_i, c_i > 0$, condition $(\mathrm{H}_1)$ also implies $b_{ii} d_{ii} > 0$ for all $i = 1, 2, \ldots, n$.