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Stability, bifurcation and a new chaos in the logistic differential equation with delay

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Abstract

This Letter is concerned with bifurcation and chaos in the logistic delay differential equation with a parameter r. The linear stability of the logistic equation is investigated by analyzing the associated characteristic transcendental equation. Based on the normal form approach and the center manifold theory, the formula for determining the direction of Hopf bifurcation and the stability of bifurcation periodic solution in the first bifurcation values is obtained. By theoretical analysis and numerical simulation, we found a new chaos in the logistic delay differential equation.

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1. Introduction

Stability, bifurcations and chaos, an interesting and complicated nonlinear phenomenon in dynamic systems, has received increasing interest during the last two decades ([1–14]). The logistic delay differential equation was advocated as adequately describing the dynamic of electrochemical intercalation and of physiological systems, etc. The logistic delay differential equation has a simple form:

$$\dot{x}(t) = -ax(t) + rx(t-\tau)\big(1 - x(t-\tau)\big), \tag{1}$$

where a is a known positive parameter, r is an unknown parameter and $\tau > 0$ is a known time delay. Notice that Eq. (1) is supplemented with an initial condition of the form

$$x(s) = \phi(s), \quad s \in [-\tau, 0].$$

A series of papers ([7,9,10]) on the application of the logistical delay differential equations have been published, but only few of them can be found on the bifurcation and chaos for the equation. This Letter is concerned with bifurcation and chaos in the logistic delay differential equation with a parameter r. The linear stability of the model is investigated by analyzing the associated characteristic transcendental equation. Using the normal form approach and the center manifold theory, we obtain the formula for determining the direction of Hopf bifurcation and the stability of bifurcation periodic solution in the first bifurcation values. By theoretical analysis and numerical simulation, we found a new chaos in the logistic delay differential equation.

This Letter is organized as follows: we perform a linear stability analysis and the existence of bifurcations of Eq. (1) in Section 2; the formulae for determining bifurcation direction and stability of the bifurcation periodic solutions of Eq. (1) are presented in Section 3, in Section 4, numerical simulations are carried out.

2. Local stability and existence of bifurcation

In this section, we first consider the local stability of the logistic delay differential equation (1).

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Notice that with the transformation $u(t) = x(\tau t)$, we can rewrite Eq. (1) as the following delay differential equation

$$\dot{u}(t) = -a\tau u(t) + r\tau u(t-1) - r\tau u^2(t-1). \tag{2}$$

The above equation has two equilibria $u^* = 0,\ 1 - \frac{a}{r}$, if $r \neq 0$, otherwise, the equation has only one $u^* = 0$.

After linearization of Eq. (2) at the neighborhood of zero, one obtains:

$$\dot{u}(t) = -a\tau u(t) + r\tau u(t-1). \tag{3}$$

The characteristic equation of Eq. (3) is of the form

$$D(\lambda) = \lambda + a\tau - r\tau e^{-\lambda} = 0. \tag{4}$$

Similarly, after linearization of Eq. (2) at the neighborhood of $u^* = 1 - \frac{a}{r},\ r \neq 0$, one gets:

$$\dot{y}(t) = -a\tau y(t) + (2a - r)\tau y(t-1), \tag{5}$$

where $y(t) = u(t) - u^*,\ u^* = 1 - \frac{a}{r}$.

The characteristic equation of Eq. (5) is of the form

$$D(\lambda) = \lambda + a\tau - (2a - r)\tau e^{-\lambda} = 0. \tag{6}$$

It is obvious that the stability of the equilibrium $u^* = 0,\ 1 - \frac{a}{r}$ depends on the roots of the characteristic equation (4), (6), respectively.

For convenience, we restate a result.

Lemma 2.1 (Hale [13]). *All roots of the characteristic equation $\lambda + c + be^{-\lambda} = 0$, where c and b are real, have negative real parts if and only if*

$$c > -1, \tag{7}$$

$$c + b > 0, \tag{8}$$

$$b < \sqrt{c^2 + \xi^2}, \tag{9}$$

where ξ is the root of $\xi = -c \tan\xi, 0 < \xi < \pi$, if $c \neq 0$ and $\xi = \pi/2$ if $c = 0$.

Applications of Lemma 2.1 to Eq. (4) with $c = a\tau,\ b = -r\tau$ and (6) with $c = a\tau,\ b = (r - 2a)\tau$ yield

Theorem 2.2. (1) *The equilibrium $u^* = 0$ of Eq. (2) is unstable if $r < r_0$ or $r > a$, where $r_0 = -\frac{1}{\tau}\sqrt{a^2\tau^2 + \xi^2}$, here ξ is the root of $\xi = -a\tau \tan\xi,\ 0 < \xi < \pi$, and local stable if $r_0 < r < a$*; (2) *The equilibrium $u^* = 1 - \frac{a}{r}$ of Eq. (2) is local stable if $a < r < 2a - r_0$, and unstable if $r < a$ or $r > 2a - r_0$.*

Now we discuss the bifurcation of Eq. (2).

Theorem 2.3. *When the parameter r passes through the critical value $r_1 = r_0 = -\frac{1}{\tau}\sqrt{a^2\tau^2 + \xi^2}$, here ξ is the root of $\xi = -a\tau \tan\xi,\ 0 < \xi < \pi$, there is a Hopf bifurcation from the equilibrium $u^* = 0$ to a periodic orbit*; *when the parameter r passes through the critical value $r_2 = a$, there is a pitchfork bifurcation from the equilibrium $u^* = 0$ to the equilibrium $u^* = 1 - \frac{a}{r}$*; *when the parameter r passes through the critical value $r_3 = 2a - r_0$, there is a Hopf bifurcation from the equilibrium $u^* = 1 - \frac{a}{r}$ to a periodic orbit.*

Proof. Suppose Eq. (4) has a pure imaginary solution $\lambda = iw_0,\ w_0 \in R^+$, for some parameter value $r = r_*$. This leads to the following equation

$$\begin{aligned} &iw_0 + a\tau - r_*\tau e^{-iw_0} \\ &\quad = (a\tau - r_*\tau\cos w_0) + (w_0 + r_*\tau \sin w_0)i = 0, \end{aligned} \tag{10}$$

which can be rewritten as

$$\begin{cases} a\tau - r_*\tau\cos w_0 = 0, \\ w_0 + r_*\tau\sin w_0 = 0. \end{cases} \tag{11}$$

So,

$$\begin{cases} r_* = \pm\frac{1}{\tau}\sqrt{a^2\tau^2 + w_0^2}, \\ w_0 = -a\tau\tan w_0. \end{cases} \tag{12}$$

By Theorem 2.2, $r_* = r_0 = -\frac{1}{\tau}\sqrt{a^2\tau^2 + w_0^2}$, where w_0 is the root of $w_0 = -a\tau\tan w_0,\ 0 < w_0 < \pi$, is the critical values of r.

The last condition for the occurrence of a Hopf bifurcation is $\frac{\mathrm{d}[\mathrm{Re}(\lambda)]}{\mathrm{d}r}|_{r=r_0} \neq 0$.

In the following, we will show that this condition is also satisfied.

Letting $\lambda = k(r) + iw(r)$ and using (4), we have

$$\begin{cases} k + a\tau - r\tau e^{-k}\cos w = 0, \\ w + r\tau e^{-k}\sin w = 0. \end{cases} \tag{13}$$

Taking the derivation of the both side of Eq. (13) with respect to r, we obtain

$$\begin{cases} \frac{\mathrm{d}k}{\mathrm{d}r} - \tau e^{-k}\cos w + r\tau e^{-k}\cos w\frac{\mathrm{d}k}{\mathrm{d}r} + r\tau e^{-k}\sin w\frac{\mathrm{d}w}{\mathrm{d}r} = 0, \\ \frac{\mathrm{d}w}{\mathrm{d}r} + \tau e^{-k}\sin w - r\tau e^{-k}\sin w\frac{\mathrm{d}k}{\mathrm{d}r} + r\tau e^{-k}\cos w\frac{\mathrm{d}w}{\mathrm{d}r} = 0. \end{cases} \tag{14}$$

Hence, we have

$$\begin{aligned} \left.\frac{\mathrm{d}[\mathrm{Re}(\lambda)]}{\mathrm{d}r}\right|_{r=r_0} &= \left.\frac{\mathrm{d}k}{\mathrm{d}r}\right|_{k=0,w=w_0,r=r_0} \\ &= \frac{\tau\cos w_0 + r_0\tau^2}{(1 + r_0\tau\cos w_0)^2 + (r_0\tau\sin w_0)^2} \\ &= \frac{a\tau + r_0^2\tau^2}{r_0[(1 + r_0\tau\cos w_0)^2 + (r_0\tau\sin w_0)^2]} \neq 0. \end{aligned} \tag{15}$$

This implies that the parameter passes through the critical value $r_1 = r_0 = -\frac{1}{\tau}\sqrt{a^2\tau^2 + \xi^2}$, where ξ is the root of $\xi = -a\tau\tan\xi,\ 0 < \xi < \pi$, there is a Hopf bifurcation from the equilibrium $u^* = 0$ to a periodic orbit.

Similarly, we can prove that the parameter r passes through the critical value $r_3 = 2a - r_0$, there is a Hopf bifurcation from the equilibrium $u^* = 1 - \frac{a}{r}$ to a periodic orbit.

Notice that $w = 0$ is always a root of Eqs. (4) and (6) if $r = a$. So the parameter r passes through the critical value $r_2 = a$, a pitchfork bifurcation occurs from the equilibrium $u^* = 0$ to the equilibrium $u^* = 1 - \frac{a}{r}$.

This complete the proof. □