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On the propagation of transient acoustic waves in isothermal bubbly liquids

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Abstract

The dynamic propagation of acoustic waves in a half-space filled with a viscous, bubbly liquid is studied under van Wijngaarden's linear theory. The exact solution to this problem, which corresponds to the compressible Stokes' 1st problem for the van Wijngaarden–Eringen equation, is obtained and analyzed using integral transform methods. Specifically, the following results are obtained: (i) van Wijngaarden's theory is found to be ill-suited to describe air bubbles in water; (ii) At start-up, the behavior of the bubbly liquid is similar to that of a class of non-Newtonian fluids under shear; (iii) Bounds on the pressure field are established; (iv) For large time, the solution exhibits Taylor shock-like (i.e., nonlinear) behavior.

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1. Introduction

Based in part on the earlier work of Lord Rayleigh [1] and Foldy [2], van Wijngaarden [3] showed in 1972 that, in the case of one spatial dimension, the propagation of linear acoustic waves in isothermal bubbly liquids, wherein the bubbles are of uniform radius, is described by the PDE

$$c_e^2\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial t^2}+\nu_e\frac{\partial^3 u}{\partial x^2\partial t}+r_0^2\frac{\partial^4 u}{\partial x^2\partial t^2}=0, \tag{1.1}$$

where $\mathbf{v}=(u(x,t),0,0)$ is the velocity vector and

$$c_e^2=\frac{p_0}{\rho_l\beta_0(1-\beta_0)},\qquad \nu_e=\frac{4\nu_l}{3\beta_0(1-\beta_0)},$$
$$r_0^2=\frac{\mathcal{R}_0^2}{3\beta_0(1-\beta_0)}. \tag{1.2}$$

Here, $\mathcal{R}_0$ (>0) is the (constant) equilibrium bubble radius; c_e and ν_e, respectively, denote the effective values of the (isothermal) sound speed and kinematic viscosity; $\nu_l=\mu_l/\rho_l$ denotes the kinematic viscosity of the liquid phase, where the constants ρ_l and μ_l $(\geqslant 0)$, respectively, denote the density and dynamic viscosity of the surrounding liquid; the constant p_0 (>0) is the equilibrium pressure in the liquid/gas mixture; and the constant β_0, where $\beta_0\in(0,1)$ is the bubble volume fraction, is neither very close to zero nor to unity.

In 1985, Caflish et al. [4] extended van Wijngaarden's theory to include heat conduction and surface tension effects. Subsequently, Eringen [5] rederived the multi-dimensional version of Eq. (1.1) based on a microcontinuum theory, and considered the case of plane waves in an unbounded, three-dimensional domain. In 1994, Saccomandi [6] investigated acoustic acceleration waves under the nonlinear version of Eringen's [5] theory. More recently, Jordan and Feuillade [7] obtained the exact solution to Eq. (1.1), which they termed the van Wijngaarden–Eringen (VWE) equation, in the context of the compressible version of Stokes' 2nd problem. For a comprehensive listing (up to 1992) of works on acoustic propagation in bubbly liquids, we note the review paper by Miksis and Ting [8]. Other recent, in-depth, works in this area include that of Llewellin et al. [9], in which a constitutive model describing the viscoelastic (i.e., non-Newtonian) rheology of bubbly liquids/suspensions is developed, and the papers by Brenner et al. [10] and Karpov et al. [11].

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It is of interest to note that Hayes and Saccomandi [12] showed that Eq. (1.1) also governs the propagation of damped, transverse plane waves in a particular class of viscoelastic solids. Additionally, it should be noted that the special case of Eq. (1.1) for which $\mu_l = 0$ and $\beta_0 \ll 1$ was presented by van Wijngaarden [13] in 1968 as the PDE governing acoustic waves in inviscid bubbly liquids. Whitham [14] noted that the inviscid version of the VWE arises in the study of plasma waves, longitudinal waves in elastic bars (see also [15,16]), and in the linear theory of water waves under the Boussinesq approximation for long waves. For applications of the $\mathcal{R}_0 \to 0$ limiting case, known as Stokes' equation, see [17] and the references therein.

To the best of our knowledge, only time-harmonic solutions of Eq. (1.1) have thus far been obtained. Hence, our aim here is to examine van Wijngaarden's theory in the context of a dynamic, yet still analytically tractable, flow setting. Specifically, we solve and analyze the compressible version of Stokes' 1st problem [18] involving the (viscous) VWE. We also derive a number of asymptotic results, including recovery of the (known) inviscid solution. To this end, the present Letter is arranged as follows. In Section 2, the exact solution to the above-mentioned initial-boundary value problem (IBVP) is obtained using integral transform methods. In Section 3, analytical results are presented including large- and small-t expressions. In Section 4 numerical results are presented and in Section 5 conclusions are stated. Lastly, in Section 6, the major results are discussed.

2. Mathematical formulation and solution

2.1. Problem formulation

We begin this study by taking the positive z-axis of a Cartesian coordinate system in the upward direction and assuming that an isothermal, homogeneous, viscous bubbly liquid fills the half-space $x > 0$. Initially, the mixture is in its equilibrium state. At time $t = 0^+$, the pressure at the boundary $x = 0$ suddenly assumes, and is maintained at, the constant value $p_{\max}$ $(\neq 0)$; i.e., the boundary condition (BC) for the pressure at $x = 0$ is $p_{\max} H(t)$, where $H(\cdot)$ denotes the Heaviside unit step function. We seek to determine the motion of the bubbly liquid at all points in the half-space for all $t > 0$.

To this end, we are lead to consider the following IBVP involving the VWE equation expressed in terms of the acoustic pressure $p = \wp - p_0$:

$$c_e^2 \frac{\partial^2 p}{\partial x^2} - \frac{\partial^2 p}{\partial t^2} + \nu_e \frac{\partial^3 p}{\partial x^2 \partial t} + r_0^2 \frac{\partial^4 p}{\partial x^2 \partial t^2} = 0 \quad (x, t > 0), \tag{2.1}$$

$$p(0,t) = p_{\max} H(t), \quad p(\infty, t) = 0 \quad (t > 0),$$
$$p(x,0) = \partial p(x,0)/\partial t = 0 \quad (x > 0). \tag{2.2}$$

Here, $\wp$ is the thermodynamic pressure, we now require $\mu_l > 0$, and we note that $\nabla \times \mathbf{v}$ is identically zero. Employing the nondimensional variables $p' = p/p_{\max}$, $x' = x(c_e/\nu_e)$, and $t' = t(c_e^2/\nu_e)$, we recast our IBVP in dimensionless form as

$$\frac{\partial^2 p}{\partial x^2} - \frac{\partial^2 p}{\partial t^2} + \frac{\partial^3 p}{\partial x^2 \partial t} + R^2 \frac{\partial^4 p}{\partial x^2 \partial t^2} = 0 \quad (x, t > 0), \tag{2.3}$$

$$p(0,t) = H(t), \quad p(\infty, t) = 0 \quad (t > 0),$$
$$p(x,0) = \partial p(x,0)/\partial t = 0 \quad (x > 0), \tag{2.4}$$

where all primes have been omitted for convenience and the dimensionless bubble radius is given by

$$R^2 \equiv \frac{c_e^2 \mathcal{R}_0^2}{3\nu_e^2 \beta_0 (1 - \beta_0)} = \left(\frac{\mathcal{R}_0 \sqrt{3 p_0 \rho_l}}{4 \mu_l} \right)^2. \tag{2.5}$$

2.2. Exact solution using integral transform methods

We will now solve the above IBVP using a dual integral transform approach (see, e.g., Duffy [19]). Hence, applying first the spatial sine transform, which reduces Eq. (2.3) to an ODE, and then using the temporal Laplace transform to solve this ODE, we obtain the dual transform domain solution

$$\bar{\hat{p}} = \frac{\xi \sqrt{2/\pi}}{1 + \xi^2 R^2} \left[\frac{1}{s(s - s_1)(s - s_2)} + \frac{R^2 s}{(s - s_1)(s - s_2)} + \frac{1}{(s - s_1)(s - s_2)} \right], \tag{2.6}$$

where ξ and s are the sine and Laplace transform parameters, respectively,

$$s_{1,2} = \frac{-\xi^2 \pm \xi \sqrt{\xi^2 (1 - 4R^2) - 4}}{2(1 + \xi^2 R^2)}, \tag{2.7}$$

and a hat (respectively, bar) superposed over a quantity denotes the image of that quantity in the sine (respectively, Laplace) transform domain.

Obtaining first the Laplace inverse of Eq. (2.6) using a table of inverses (see, e.g., [19,20]), multiplying the result by $\sqrt{2/\pi} \sin[\xi x]$, and then integrating with respect to ξ from zero to infinity, we find the exact xt-domain solution to be

$$\begin{aligned}
p(x,t) &= H(t) \Biggl\{ 1 - \frac{2}{\pi} \Biggl[\int_0^{\xi^*} \mathrm{e}^{-at} \left(\cos[bt] + \frac{a}{b} \sin[bt] \right) \frac{\sin[\xi x]\, \mathrm{d}\xi}{\xi} \\
&\quad + \int_{\xi^*}^{\infty} \mathrm{e}^{-at} \left(\cosh[bt] + \frac{a}{b} \sinh[bt] \right) \frac{\sin[\xi x]\, \mathrm{d}\xi}{\xi} \\
&\quad - \left[\int_0^{\xi^*} \frac{\xi \mathrm{e}^{-at} \sin[bt]}{b} \frac{\sin[\xi x]\, \mathrm{d}\xi}{1 + \xi^2 R^2} + \int_{\xi^*}^{\infty} \frac{\xi \mathrm{e}^{-at} \sinh[bt]}{b} \frac{\sin[\xi x]\, \mathrm{d}\xi}{1 + \xi^2 R^2} \right] \\
&\quad - R^2 \Biggl[\int_0^{\xi^*} \xi \mathrm{e}^{-at} \left(\cos[bt] - \frac{a}{b} \sin[bt] \right) \frac{\sin[\xi x]\, \mathrm{d}\xi}{1 + \xi^2 R^2}
\end{aligned}$$