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Physics Letters A 350 (2006) 75–80
PHYSICS LETTERS A
www.elsevier.com/locate/pla

Integrability of invariant metrics on the Virasoro group

A. Constantin [a,c,*], B. Kolev [b], J. Lenells [c]

[a] Trinity College, School of Mathematics, Dublin 2, Ireland
[b] CMI, 39, rue F. Joliot-Curie, 13453 Marseille cedex 13, France
[c] Department of Mathematics, Lund University, 22100 Lund, Sweden

Received 22 April 2005; received in revised form 4 October 2005; accepted 5 October 2005

Available online 11 October 2005

Communicated by A.P. Fordy

Abstract

Each right-invariant metric on the Virasoro group induces a Hamiltonian vector field on the dual of the Lie algebra $\mathfrak{vir}$ equipped with the canonical Lie–Poisson structure. We show that the Hamiltonian vector fields X_k induced by the metrics given at the identity by the H^k Sobolev inner products, $k \geqslant 0$, are bi-Hamiltonian relative to a modified Lie–Poisson structure only for $k = 0$ and $k = 1$.

MSC: 35Q35; 37K45

Keywords: Bi-Hamiltonian structures; Virasoro group

1. Introduction

The well-known Korteweg–de Vries [17] and Camassa–Holm [2,9] shallow water equations can be viewed as Euler equations on the dual of the Lie algebra of the Virasoro group. Indeed, the periodic Korteweg–de Vries equation represents geodesics on the Virasoro group for the right-invariant metric given at the identity by the L^2 inner product [19], whereas the Camassa–Holm equation describes the geodesic flow of the right-invariant metric given at the identity by the H^1 inner product (see [5,6,18]). While all solutions of the Korteweg–de Vries equation are global [14], there are solutions of the Camassa–Holm equation with finite and also solutions with infinite maximal existence time [3]. Since both these equations have a rich structure (they are both integrable [8,16,17]), it is natural to ask whether the equations induced on the Virasoro group by the right-invariant H^k Sobolev inner products for $k \geqslant 2$ possess similar properties.

One aspect of the Korteweg–de Vries and Camassa–Holm equations that could be expected to carry over to all H^k metrics is the existence of a bi-Hamiltonian structure. The Euler equation on the dual $\mathfrak{g}^*$ of a Lie algebra is always Hamiltonian with respect to the canonical Lie–Poisson structure [1]. In some cases there is another Poisson structure on $\mathfrak{g}^*$ such that the equation is Hamiltonian with respect to this second structure as well. The Korteweg–de Vries and Camassa–Holm equations are bi-Hamiltonian with respect to so-called modified Lie–Poisson structures, leading to integrability and the existence of an infinite number of conservation laws for these equations.

In this Letter we show that on the Virasoro group the Euler equations corresponding to the H^k inner products are not bi-Hamiltonian for $k \geqslant 2$, at least not with respect to a modified Lie–Poisson structure. A similar result was proved for the diffeomorphism group of the circle $\mathcal{D}(\mathbb{S})$ in [7], where it was found that among all H^k inner products, $k \geqslant 0$, only in the cases $k = 0$ and $k = 1$ is the geodesic equation bi-Hamiltonian with respect to a modified Lie–Poisson structure.

* Corresponding author.
E-mail addresses: adrian@maths.tcd.ie, adrian.constantin@math.lu.se (A. Constantin), boris.kolev@up.univ-mrs.fr (B. Kolev), jonatan@maths.lth.se (J. Lenells).

0375-9601/$ – see front matter
doi:10.1016/j.physleta.2005.10.011

Since the Virasoro group is a one-dimensional central extension of $\mathcal{D}(\mathbb{S})$, one might think that the cases of $\mathcal{D}(\mathbb{S})$ and the Virasoro group can be reduced to one another. However, this intuitive idea has to be analyzed with caution. For example, the geodesic exponential map on $\mathcal{D}(\mathbb{S})$ corresponding to the right-invariant H^k Sobolev metric is a C^1-diffeomorphism from a neighborhood of the origin of the Lie algebra Vect($\mathbb{S}$) to a neighborhood of the unital element in $\mathcal{D}(\mathbb{S})$ for all $k \geqslant 1$, but not for $k = 0$ [6]. On the Virasoro group, however, the geodesic exponential map is a local C^1-diffeomorphism for $k \geqslant 2$, but not for $k = 0, 1$ [4].

In Section 2 we recall some basic facts concerning the Virasoro group and its Lie algebra. Section 3 contains a description of Poisson structures on the dual of the Lie algebra of the Virasoro group. Finally, the main result is stated and proved in Section 4.

2. Preliminaries

Let $\mathbb{S}$ be the unit circle. The group $\mathcal{D}(\mathbb{S})$ of positively oriented C^∞-diffeomorphisms of the circle is a Fréchet manifold modeled on the space $C^\infty(\mathbb{S})$ (see [4] for a more detailed description of this and other definitions made in this section).

The Virasoro group Vir is the central extension $\mathcal{D}(\mathbb{S}) \times \mathbb{R}$ of $\mathcal{D}(\mathbb{S})$ with group multiplication given by the formula

$$(\phi, \alpha) \circ (\psi, \beta) = \left(\phi \circ \psi, \alpha + \beta - \frac{1}{2} \int_{\mathbb{S}} \log\bigl((\phi \circ \psi)_x(x)\bigr)\, d \log \psi_x(x)\right),$$

where $(\phi, \alpha), (\psi, \beta) \in \mathrm{Vir}$, see [12].

Vir is a Fréchet Lie group whose Lie algebra $\mathfrak{vir}$ can be identified with the space $C^\infty(\mathbb{S}) \times \mathbb{R}$ with Lie bracket

$$\bigl[(u, a), (v, b)\bigr] = \left(u_x v - u v_x, \int_{\mathbb{S}} u v_{xxx}\, dx\right),$$

where $(u, a), (u, b) \in \mathfrak{vir} \simeq C^\infty(\mathbb{S}) \times \mathbb{R}$.

We will let $\mathfrak{vir}^*$ denote the *regular dual* of $\mathfrak{vir}$ consisting of all linear functionals on $\mathfrak{vir}$ of the form

$$(u, a) \mapsto \int_{\mathbb{S}} m u\, dx + \alpha a, \quad (u, a) \in \mathfrak{vir},$$

for some element $(m, \alpha) \in C^\infty(\mathbb{S}) \times \mathbb{R}$. The regular dual $\mathfrak{vir}^*$ is identified with the Fréchet space $C^\infty(\mathbb{S}) \times \mathbb{R}$ by the formula

$$\bigl\langle (u, a), (m, \alpha) \bigr\rangle = \int_{\mathbb{S}} m u\, dx + \alpha a,$$

where $\langle \cdot, \cdot \rangle$ denotes the natural pairing between $\mathfrak{vir}$ and $\mathfrak{vir}^*$. For $k \geqslant 0$ let us define the H^k Sobolev inner product on $\mathfrak{vir}$ by

$$\bigl\langle (u, a), (v, b) \bigr\rangle_k = \int_{\mathbb{S}} A_k(u) v\, dx + ab = \int_{\mathbb{S}} \sum_{i=0}^{k} \bigl(\partial_x^i u\bigr)\bigl(\partial_x^i v\bigr)\, dx + ab,$$

where

$$A_k = 1 - \frac{d^2}{dx^2} + \cdots + (-1)^k \frac{d^{2k}}{dx^{2k}}.$$

Let $f : \mathfrak{vir}^* \to \mathbb{R}$ be a smooth function. Its Fréchet derivative at a point $M \in \mathfrak{vir}^*$, $df(M)$, is a linear functional on $\mathfrak{vir}^*$. We call f a *regular function* if there exists a smooth map $\delta f : \mathfrak{vir}^* \to \mathfrak{vir}$ such that

$$df(M)N = \bigl\langle \delta f(M), N \bigr\rangle, \quad M, N \in \mathfrak{vir}^*.$$

Note that the second Fréchet derivative of a function $f \in C^\infty(\mathfrak{vir}^*)$ is symmetric [13], i.e.,

$$d^2 f(M)(N, P) = d^2 f(M)(P, N), \quad M, N, P \in \mathfrak{vir}^*.$$

This means that $d\delta f(M) : \mathfrak{vir}^* \to \mathfrak{vir}$ is a symmetric operator

$$\bigl\langle d\delta f(M) N, P \bigr\rangle = \bigl\langle d\delta f(M) P, N \bigr\rangle, \quad M, N, P \in \mathfrak{vir}^*.$$

The variational derivative δf can be represented by a map

$$\left(\frac{\delta f}{\delta m}, \frac{\partial f}{\partial \alpha}\right) : \mathfrak{vir}^* \to \mathfrak{vir} \simeq C^\infty(\mathbb{S}) \times \mathbb{R}$$

so that

$$df(M)N = \left\langle \left(\frac{\delta f}{\delta m}(M), \frac{\partial f}{\partial \alpha}(M)\right), N \right\rangle, \quad M, N \in \mathfrak{vir}^*.$$