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How additive noise generates a phantom attractor in a model with cubic nonlinearity

Irina Bashkirtseva, Lev Ryashko *

Ural Federal University, Pr. Lenina 51, Ekaterinburg, 620000, Russia

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ABSTRACT

Two-dimensional nonlinear system forced by the additive noise is studied. We show that an increasing noise shifts random states and localizes them in a zone far from deterministic attractors. This phenomenon of the generation of the new "phantom" attractor is investigated on the base of probability density functions, mean values and variances of random states. We show that increasing noise results in the qualitative changes of the form of pdf, sharp shifts of mean values, and spikes of the variance. To clarify this phenomenon mathematically, we use the fast–slow decomposition and averaging over the fast variable. For the dynamics of the mean value of the slow variable, a deterministic equation is derived. It is shown that equilibria and the saddle-node bifurcation point of this deterministic equation well describe the stochastic phenomenon of "phantom" attractor in the initial two-dimensional stochastic system.

1. Introduction

It is well known that any real dynamic system is influenced by the random noise. Even small stochastic disturbances can cause unexpected outcomes, especially in nonlinear systems [1,2]. In the modern stochastic dynamics, a great variety of noise-induced phenomena related to the stochastic resonance [3–5], noise-induced transitions between order and chaos [6–8], stochastic excitability [9], noise-induced annihilation of attractors [10], are studied. From the mathematical point of view, many of phenomena are treated as stochastic bifurcations. Here, a notion of P-bifurcation (phenomenological) [11] is commonly used. These bifurcations are defined as qualitative deformations of the form of the stationary probability density function (pdf) in a response to the change of the noise intensity. There are many examples where noise changes a number of peaks, and causes an appearance/disappearance of craters and ridges in the form of pdf [12–20].

Traditionally, in studies of stochastic systems, an influence of the additive and parametric noise is compared and frequently opposed. It is well known that under the variation of the functional dependence of the noise intensity on the system state, almost any shape of pdf can be obtained. Results of the influence of additive noise with constant intensity are not so various. For example, in one-dimensional systems, extrema of pdf are exactly located above equilibria. An additive noise changes only the dispersion of random states near stable equilibria.

In two-dimensional systems, an additive noise can generate more rich variety of pdf-forms. There are examples of systems with a single stable equilibrium, where additive noise results not only in the one peak above this equilibrium but also forms additional craters and ridges. FitzHugh–Nagumo system [21–23] is a classical model of this type. A recent example reported in [24] demonstrates how additive noise in the system with two stable and one unstable equilibria can generate pdf with a single peak above the unstable equilibrium. All these stochastic phenomena are due to high non-homogeneity of initial deterministic phase portraits.

In present paper, we consider a two-parametric 2D-system with cubic nonlinearity [25]. In the sufficiently narrow parametric zone, this system exhibits changes of deterministic attractors: a single stable equilibrium – two stable equilibria – two stable cycles. This system attracts attention because of the unexpected reply on the simple Gaussian additive noise. In this system, noise abruptly changes a shape of pdf, and transforms it to the unimodal one with a single peak. This form of pdf is typical for a stochastically forced equilibrium. However, the system under consideration has no any attractor in this zone of the phase plane. So, we call this unexpected phenomenon by "phantom" stochastic attractor.

The aim of the present paper is to study this noise-induced phenomenon and explain its probabilistic mechanism on the base of the analysis of peculiarities of the initial deterministic phase portrait. In this study, we use a direct numerical simulation and

* Corresponding author.
E-mail address: lev.ryashko@urfu.ru (L. Ryashko).

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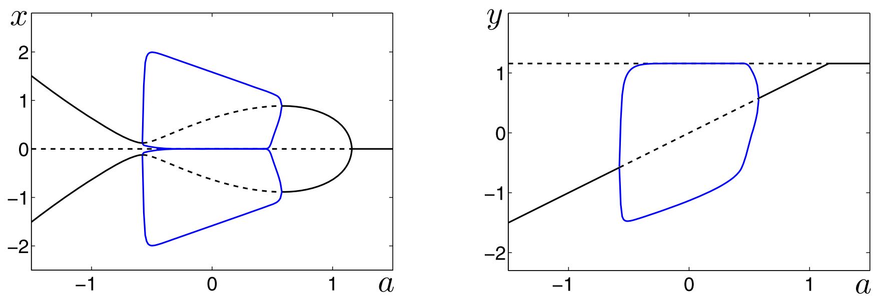

Fig. 1. Bifurcation diagram of system (1). (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

analytical approach based on the decomposition of fast–slow variables and method of averaging over the fast variable.

In Section 2, qualitative changes of the deterministic dynamics of the system with cubic nonlinearity under the variation of the parameter are shown, and corresponding bifurcations are discussed.

In Section 3, we study an influence of the additive noise on this system by direct numerical simulation of its solutions. To describe shifts of zones where random states are concentrated, probability density functions are used. These functions register an appearance of the "phantom" stochastic attractor. For the parametric study of these transformations of pdf, mean values and variances are analyzed.

In Section 4, an analytical approach is suggested to explain a probabilistic mechanism of the stochastic phenomena under consideration. Here, the fast–slow decomposition of the system variables is given, and on the base of the averaging over the fast variable, one-dimensional equation is derived. We show that equilibria and their bifurcations in this simple deterministic equation throw light on the reasons of the appearance of "phantom" attractor in the initial two-dimensional stochastic system.

2. Analysis of deterministic system

Consider a system of two nonlinear differential equations

$$\begin{cases} \dot{x} = (y-a)x, \\ \dot{y} = \mu + y - y^3 - x^2. \end{cases} \tag{1}$$

This system connected with planar truncated cusp-Hopf normal form was considered in [25].

It follows from the system (1) that the variable y is governed by the second-order differential equation

$$\ddot{y} + p(y)\dot{y} + q(y) = 0,$$

where $p(y) = 3y^2 - 2(y-a) - 1$, $q(y) = 2(y-a)(\mu + y - y^3)$. Thus, a behavior of the system (1) is directly connected with the dynamics of this nonlinear oscillator. Parameters of this oscillator have a standard physical sense: $p(y)$ is a nonlinear dissipation, $q(y)$ defines a form of the fifth-order potential

$$U(y) = \int_0^y q(s)ds = -\frac{2}{5}y^5 + \frac{a}{2}y^4 + \frac{2}{3}y^3 + (\mu - a)y^2 - \frac{a}{2}\mu y.$$

Considering the Jacobian matrix of the system (1)

$$F = \begin{pmatrix} y-a & x \\ -2x & 1-3y^2 \end{pmatrix},$$

we have $\mathrm{tr}F = 1 - a + y - 3y^2$, $\det F = (y-a)(1-3y^2) + 2x^2$.

In what follows, we fix $\mu = 0.4$. In this case, the system (1) has equilibria $M_0(0, \hat{y})$ and $M_{1,2}(\pm\sqrt{0.4 + a - a^3}, a)$. The value $\hat{y} \approx 1.1597$ is a unique solution of the equation $0.4 + y - y^3 = 0$. Note that equilibria $M_{1,2}$ exist only for $a \le \hat{y}$.

At the point M_0, we have $\mathrm{tr}F = -1.875 - a$, $\det F = 3.0347(a - \hat{y})$. For the equilibria $M_{1,2}$, we get $\mathrm{tr}F = 1 - 3a^2$, $\det F = 2(0.4 + a - a^3)$.

So, one can mark three bifurcation points: $a_1 = -\sqrt{3}/3$, $a_2 = \sqrt{3}/3$, $a_3 = \hat{y} = 1.1597$.

The equilibrium M_0 is stable for $a > a_3$, and unstable otherwise. Points $M_{1,2}$ are stable for $a < a_1$ and for $a_2 < a < a_3$, and unstable for $a_1 < a < a_2$. Here, a_3 is a pitchfork bifurcation point, and $a_{1,2}$ are Andronov–Hopf bifurcation points. For $a_1 < a < a_2$, the system possesses two stable cycles. Attractors and repellers of system (1) are plotted in Fig. 1 where shown are stable equilibria (black solid), unstable equilibria (black dashed), and extrema of coordinates of limit cycles (blue solid).

In Fig. 2, phase portraits of system (1) are presented for three values of the parameter a. For $a = 0.5$, two stable cycles coexist (see red curves in Fig. 2a). For $a = 0.6$, system (1) is bistable with two stable equilibria M_1 and M_2. Note that in the case of bistability, basins of attraction of symmetric attractors are separated by the line $x = 0$. This line is a stable manifold of the saddle point M_0. For $a = 1.2$, system (1) is monostable with the single stable equilibrium M_0.

3. Noise-induced transitions in stochastic system

Consider system (1) under the influence of the additive noise

$$\begin{cases} \dot{x} = (y-a)x + \varepsilon\xi \\ \dot{y} = \mu + y - y^3 - x^2. \end{cases} \tag{2}$$

Here, ε is a noise intensity, and ξ is a standard Gaussian white noise with parameters $\mathrm{E}\xi(t) = 0$, $\mathrm{E}\xi(t)\xi(\tau) = \delta(t-\tau)$.

Consider an influence of noise on the attractors of system (1). For $\mu = 0.4$ and three values of the parameter a, random states of system (2) and corresponding probability density functions (pdf) are shown for various values of the noise intensity in Figs. 3, 4, 5.

As one can see, for weak noise, random states are concentrated near deterministic attractors. This fact is accompanied by the one-peak or crater-like form of pdf above the stable equilibria or limit cycles, correspondingly. For example, for $a = 0.5$, there are two craters above two stable limit cycles of the initial deterministic system (see Fig. 3 for the noise intensity $\varepsilon = 0.02$). For $a = 0.6$, the deterministic system possesses two stable equilibria, and under small noise pdf has two peaks (see Fig. 4 for the noise intensity $\varepsilon = 0.05$). For $a = 1.2$, one peak of pdf appears above the single stable equilibrium (see Fig. 5 for the noise intensity $\varepsilon = 0.05$).

With increasing noise, probability density functions change. Here, two phenomena should be mentioned.

The first of them is quite obvious: increasing noise smoothes out peaks and craters. In bistability cases (see Figs. 3, 4), random trajectories can cross the separatrix $x = 0$ and exhibit oscillations