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Dirac equation on a curved surface

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ABSTRACT

The dynamics of Dirac particles confined to a curved surface is examined employing the *thin-layer method*. We perform a perturbative expansion to first-order and split the Dirac field into normal and tangential components to the surface. In contrast to the known behavior of second order equations like Schrödinger, Maxwell and Klein–Gordon, we find that there is no geometric potential for the Dirac equation on a surface. This implies that the non-relativistic limit does not commute with the thin-layer method. Although this problem can be overcome when second-order terms are retained in the perturbative expansion, this would preclude the decoupling of the normal and tangential degrees of freedom. Therefore, we propose to introduce a first-order term which rescues the non-relativistic limit and also clarifies the effect of the intrinsic and extrinsic curvatures on the dynamics of the Dirac particles.

1. Introduction

The behavior of quantum systems confined on a surface has been extensively investigated for decades in the context of condensed matter physics. Although the main focus has been on non-relativistic systems, recently there has been renewed interest in systems that have relativistic dispersion relations at low energy, such as graphene and topological insulators. It has been shown that a flat sample of graphene can be modeled by a massless Dirac Hamiltonian which can be derived from the tight-binding model in the continuum approximation [1]. If the sample of graphene presents curvature, there are two main approaches to model it [2,3]. One model is based on a combination of tight-binding and elasticity theory [4] and the other model is based on the formulation of quantum field theory in curved space [5]. In the latter, it is assumed that the dynamics of carriers in graphene in low energies is described by Dirac equation in $(2+1)$ dimensional curved space–time. More recently, however, Atanasov and Saxena have pointed out that, to be consistent with the Heisenberg uncertainty principle, these approaches should not overlook the fact that the carriers are intrinsically three-dimensional objects [6].

Already in the seventies Jensen and Koppe [7] and independently Costa ten years later [8], developed a formalism with the aim of studying the Schrödinger equation confined to a curved surface. The so-called *thin-layer quantization* (a confining potential formalism) takes into account the full three-dimensional character of the carriers. Unlike previous methods, it constitutes a well-defined procedure which does not suffer from the well-known ordering ambiguity problem, and it is consistent with the Heisenberg's uncertainty principle. In this approach, the Schrödinger equation on the surface can be derived and the result contains a *geometrical potential* expressed in terms of both the intrinsic and the *extrinsic curvatures* [7,8]. It has been shown that for flat surfaces the resulting Schrödinger equation is simply the Schrödinger equation for two-dimensional systems. It is for this reason that this equation has been so effective to describe quasi-two-dimensional systems. However, there are significant effects in the presence of curvature, such as the geometrical potential, which has been the subject of intensive research [9–16].

The remarkable feature of the thin-layer approach is that it assumes that there is a physical mechanism that confines the particles on the surface. Three conditions are required for the confining *physical* potential V_ϵ [7,8,17]: (i) V_ϵ has a deep minimum relative to the surface; (ii) V_ϵ depends only on coordinates, which are normal to the surface and (iii) V_ϵ depends the parameter ϵ in such a way that as $\epsilon \to 0$ the potential goes to infinity outside of the surface. If the confinement occurs for example along the z-direction, the parameter ϵ is related to the confinement width (l) through $l^2 = \langle z^2 \rangle = \epsilon^2 l_0^2$, where l_0^2 is a characteristic length. Thus, ϵ is a measure of how much the system can be modeled by a quasi-two-dimensional system.

Since from the experimental point of view it is *unrealistic* take $\epsilon \to 0$ (and incompatible with the Heisenberg uncertainty principle), it is more appropriate to consider a perturbative expansion in ϵ which can be understood as an *effective theory* on the surface. In

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the non-relativistic case, a complete perturbative expansion of the Hamiltonian was performed in Ref. [17]. The zeroth-order term can be interpreted as strong fluctuations of the system in directions normal to the surface. The next non-vanishing term can be interpreted as the Hamiltonian describing the effective dynamics on a surface [7,8,17]. Higher-order terms take into account further couplings between normal and tangent degrees of freedom [17,18]. The effective Schrödinger equation on a surface is given by [7,8,17]

$$i\hbar\frac{\partial\psi}{\partial t}=\frac{-\hbar^2}{2m|g|^{1/2}}\partial_i g^{ij}|g|^{1/2}\partial_j\psi-\frac{\hbar^2}{2m}(H^2-K)\psi, \tag{1}$$

where g_{ij} is the metric on the surface, H is the *mean curvature* (extrinsic) and K is the *Gaussian curvature* (intrinsic). The last term in Eq. (1) is a curvature-induced quantum potential, *i.e.*, the so-called geometrical potential.

The thin-layer method has been extended to include external electromagnetic fields [19,13,20,21], generalized to codimensions higher than one [22–24,17,25] and has also been applied to the study of Maxwell [26,27], Pauli [28,29], Klein–Gordon [30,31] and Dirac [32–38,6,39] equations. Furthermore, in 2010, Szameit et al. found experimentally an optical analogue of the geometric potential [27] and Onoe et al. reported in 2012 the first experimental evidence of the geometric potential in a quantum system [40].

The effective Dirac equation on a surface was considered long ago by Burgess and Jensen [33], for surfaces with zero intrinsic curvature, and it has been used in the following form for more than two decades[1] [32–37,6,39,41]

$$\left[i\hbar e_a^\mu\gamma^a\left(\partial_\mu-\tilde{\Omega}_\mu\right)-mc+\gamma^3 H\right]\chi=0, \tag{2}$$

where the e_a^μ are the dreibeins and the γ^a are the Dirac matrices. Although it is sometimes overlooked, we point out that the original deduction in Ref. [33] was performed only for surfaces with zero Gaussian curvature. As we will show it is not clear from previous investigations [32–37,6,39] what would be the exact meaning of $\tilde{\Omega}_\mu$. This has led to the misconception that there is a *geometric potential* for the Dirac equation in a perturbative expansion up to first-order in ϵ. An exception is found in the article of Maraner and Pachos, [38], where the effective Dirac equation on the surface, up to first-order in ϵ, appears without the geometric potential. However, the authors do not acknowledge any disagreement with the previous literature, nor show the explicit calculations that would explain why the geometric potential is absent.

In this paper, we perform a detailed derivation of the Dirac equation on a surface using the thin-layer approach. As we shall see, contrary to the usual belief, there is no geometric potential for the Dirac equation to first-order in ϵ. This is in contrast with the Schrödinger, Pauli, Maxwell and Klein–Gordon equations on a surface, where a geometric potential is generated to first order in ϵ. The absence of geometric potential in the Dirac equation implies that the non-relativistic limit may not commute with the thin-layer procedure. However, this can be understood once we realize that the geometrical potential of second order equations emerges from second-order terms in ϵ and that when taking the non-relativistic limit after implementing the thin-layer method, some second-order terms in ϵ are lost. Therefore, we show that it is possible to add a term to the effective Dirac equation on the surface, such that the non-relativistic limit is recovered and some consistency conditions are satisfied.

Our paper is organized as follows. In Sec. 2 we introduce non-coordinate bases for an adapted coordinate frame. In Sec. 3 we derive the effective Dirac equation for a particle constrained to a surface. In Sec. 4 we discuss the non-relativistic limit of the effective Dirac equation on a surface. Finally, Sec. 5 contains our conclusions.

2. Non-coordinate bases for a surface

Consider a surface S embedded in a three-dimensional Euclidean space. A coordinate transformation from the Cartesian coordinate system (X^1, X^2, X^3), to an *adapted* curvilinear coordinates system (q^1, q^2, q^3), in a sufficiently small neighborhood of S, is given by

$$\mathbf{X}(q^1,q^2,q^3)=\mathbf{x}(q^1,q^2)+q^3\mathbf{n}(q^1,q^2), \tag{3}$$

where $\mathbf{x}(q^1,q^2)$ are the Cartesian coordinates of a point in S, $\mathbf{n}(q^1,q^2)$ is an orthonormal vector field to S, (q^1,q^2) are coordinates parametrizing the surface and $|q^3|$ is the distance between S and the point of coordinates (q^1,q^2,q^3). The components of the metric tensor are defined by

$$G_{MN}=\frac{\partial\mathbf{X}}{\partial q^M}\cdot\frac{\partial\mathbf{X}}{\partial q^N}, \tag{4}$$

where the Latin indices $M, N, \ldots$ run from 1 to 3. One can completely characterize the surface by the first fundamental form $g_{ij}\equiv\mathbf{t}_i\cdot\mathbf{t}_j$ (the metric) and by the second fundamental forms, the so-called *extrinsic curvature*, $\alpha_{3ij}\equiv-\mathbf{t}_i\cdot\partial_j\hat{\mathbf{n}}$, where $\mathbf{t}_i$ are two tangent vectors to the surface[2] [42]. Additionally, a *space–time* point can be described using the time coordinate $ct=X^0=q^0$. Thus, one can construct a metric for a $(3+1)$-dimensional Minkowskian space–time in the adapted coordinate system as

$$ds^2=G_{AB}dq^Adq^B=G_{00}dq^0dq^0-G_{MN}dq^Mdq^N, \tag{5}$$

where $G_{00}=1$, the Latin indices $A, B, \ldots$ run from 0 to 3. Note that the space–time is flat and in the coordinates X^A the metric is $ds^2=\eta_{AB}dX^AdX^B$, with $\eta_{CD}=diag(+1,-1,-1,-1)$.

Thus, one can rewrite G_{AB} in the adapted coordinates system as (the Greek indices $\mu, \nu, \ldots$ run from 0 to 2)

$$G_{AB}=\begin{pmatrix}\gamma_{\mu\nu} & 0\\ 0 & \eta_{33}\end{pmatrix}, \tag{6}$$

with[3] $\eta_{33}=\eta^{33}=-1$ and where

$$\gamma_{\mu\nu}=g_{\mu\nu}-2q^3\alpha_{3\mu\nu}+(q^3)^2\alpha_{3\mu\rho}g^{\rho\sigma}\alpha_{3\sigma\nu}, \tag{7}$$

where $g_{\mu\nu}$ is defined in the hypersurface by

$$g_{\mu\nu}dq^\mu dq^\nu=dq^0dq^0-g_{ij}dq^idq^j, \tag{8}$$

with

$$g_{ij}=\frac{\partial\mathbf{X}}{\partial q^i}\cdot\frac{\partial\mathbf{X}}{\partial q^j} \tag{9}$$

and $\alpha_{3\mu\nu}$ is defined in the hypersurface by

$$\alpha_{3\mu\nu}=\begin{pmatrix}0 & 0\\ 0 & \alpha_{3ij}\end{pmatrix}. \tag{10}$$

It is clear from the definitions that $g_{\mu\nu}$ and $\alpha_{3\mu\nu}$ do not depend on q^0.

Let us introduce a *non-coordinate basis*, $\hat{E}_I$, which is obtained by a rotation of a *coordinate basis*, $\{E_A\}=\{\partial/\partial_A\}$, preserving the orientation,[4]

[1] The sign of the mean curvature depends on the choice of the unit normal vector and has no intrinsic meaning, Eq. (2) is not necessarily invariant to the choice of the normal vector, which leads to ambiguity.

[2] Latin indices $i, j, \ldots$ run from 1 to 2.

[3] In our conventions $\alpha_{3\mu\rho}g^{\sigma\rho}=-\alpha_{3\mu}{}^{\rho}$.

[4] Latin indices $I, J, \ldots$, run from 0 to 3.