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Conditions for monogamy of quantum correlations in multipartite systems

Asutosh Kumar

Harish-Chandra Research Institute, Chhatnag Road, Jhunsi, Allahabad 211 019, India

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ABSTRACT

Monogamy of quantum correlations is a vibrant area of research because of its potential applications in several areas in quantum information ranging from quantum cryptography to co-operative phenomena in many-body physics. In this paper, we investigate conditions under which monogamy is preserved for functions of quantum correlation measures. We prove that a monogamous measure remains monogamous on raising its power, and a non-monogamous measure remains non-monogamous on lowering its power. We also prove that monogamy of a convex quantum correlation measure for arbitrary multipartite pure quantum state leads to its monogamy for mixed states in the same Hilbert space. Monogamy of squared negativity for mixed states and that of entanglement of formation follow as corollaries of our results.

1. Introduction

Quantum correlations [1,2], of both entanglement [1] and information-theoretic [2] paradigms, is an indispensable resource in quantum information theory [3]. While entanglement measures capture the nonseparability of two or more subsystems, information-theoretic measures like quantum discord [4,5] can detect nonclassical properties even in separable states. It is desirable that a quantum correlation measure $\mathcal{Q}$ belonging to either of two above classes satisfies certain basic properties [1,2,6] such as *positivity*, $\mathcal{Q}(\rho_{AB}) \geq 0$, and *monotonicity*, i.e., is non-increasing under a suitable set of local quantum operations and classical communications [in particular, invariance under local unitaries $U_A \otimes V_B$, $\mathcal{Q}(\rho_{AB}) = \mathcal{Q}(U_A \otimes V_B \rho_{AB} U_A^\dagger \otimes V_B^\dagger)$, as well as *no-increase upon attaching a local pure ancilla*, $\mathcal{Q}(\rho_{AB}) \geq \mathcal{Q}(\rho_{AB} \otimes |0\rangle_C\langle 0|)$]. These properties are valid for several known measures of quantum correlations, including all entanglement measures. In particular, positivity and invariance under local unitaries are standard requirements [7].

Quantum correlations, entanglement in particular, is crucial in quantum information processing and quantum computation [3], in describing area laws [8–19], in quantum phase transition and detecting other cooperative quantum phenomena in various interacting quantum many-body systems [20–23]. Hence, quantum correlations form a fundamental aspect of modern physics and a key enabler in quantum communication and computation technologies. Being a resource, quantification of quantum correlations is important. Although a number of correlation measures for bipartite (qubit) systems have been studied extensively in last few decades, there has not been much investigation of multipartite correlations owing to difficulty in defining multipartite correlations.

The concept of monogamy [24,25] is a distinguishing feature of quantum correlations, which sets it apart from classical correlations. Monogamy of quantum correlations is an active area of research, and has found potential applications in quantum information theory like in quantum key distribution [26–28], in classifying quantum states [29–31], in distinguishing orthonormal quantum bases [32], in black-hole physics [33,34], to study frustrated spin systems [35], etc. Moreover, it has proved to be a useful tool in exploring multipartite nonclassical correlations [24,25,36]. Qualitatively, monogamy of quantum correlations places certain restrictions on distribution of quantum correlations of one fixed party with other parties of a multipartite system. In particular, *if party A in a tripartite system ABC is maximally quantum correlated with party B, then A cannot be correlated at all to the third party C.* This is true for all quantum correlation measures, and is a departure from classical correlations which are not bound to such constraints. That is, classical correlations do not satisfy a monogamy constraint [37–46]. In other words, monogamy forbids free sharing of quantum correlations among the constituents of a multipartite quantum system. This is a nonclassical property in the sense that such constraints are not observed even in the maximally classically-correlated systems like

$$\rho_{ABC} = \frac{1}{2}\left(|000\rangle\langle 000| + |111\rangle\langle 111|\right). \tag{1}$$

E-mail address: asukumar@hri.res.in.

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However, two or more parties in a multipartite quantum state do not necessarily always share maximal quantum correlation, and are thus able to share some correlations with other parties, although in a restrictive manner. Thus, monogamy relations help in determining entanglement structure in the multipartite setting. Furthermore, it has been argued to be a consequence of the no-cloning theorem [47–49]. Monogamy, like entanglement [50], appears to be the trait of multipartite entangled quantum systems. Interestingly, the notion of monogamy is not restricted only to quantum correlation measures, but has spawned its wing in other quantum properties such as Bell inequality [51–53], quantum steering [54], and contextual inequalities [55–57]. A quantum correlation measure that satisfies the "monogamy inequality" for all quantum states is termed "monogamous". However, we know that not all quantum correlation measures, even for three-qubit states, satisfy monogamy. Entanglement measures such as concurrence [58,59], entanglement of formation [60], negativity [61], etc., apart from information-theoretic measures such as quantum discord [4,5] are known to be, in general, non-monogamous. In recent developments on monogamy, we have seen that exponent of a quantum correlation measure and multipartite quantum states play a remarkable role in characterization of monogamy [62,63]. A non-monogamous quantum correlation measure can become monogamous, for three or more parties, when its power is increased [62]. For instance, concurrence, entanglement of formation, negativity, quantum discord are non-monogamous for three-qubit states, but their squared versions are monogamous. In particular, it has been shown that *monotonically increasing functions of any quantum correlation can make all multiparty states monogamous* with respect to that measure [62]. We note that *the increasing function of the correlation measure under consideration satisfies all the necessary properties for being a quantum correlation measure including positivity and monotonicity under local operations*, mentioned above. Furthermore, the function can be so chosen that it is reversible [64,65], such that the information about quantum correlation in the state under consideration, after applying the function on the quantum correlation remains intact. The power of a correlation measure is an example of such a function. It is interesting to note that the function $f(x) = x^\alpha$ is *concave* for $0 < \alpha \le 1$ and *convex* for $1 \le \alpha \le \infty$ on the interval $(0, \infty)$. The power function has an intrinsic geometric interpretation. The power defines the slope of the graph. The higher power, the graph is nearer to the vertical axis. It has been found that several measures of quantum correlations like squared concurrence [24,25], squared negativity [66–68], squared quantum discord [36], global quantum discord [69,70], squared entanglement of formation [71,72], Bell inequality [73–75], EPR steering [76,77], contextual inequalities [78,79], etc. exhibit monogamy property. Thus, we observe that the convexity plays a key role in establishing monogamy of quantum correlations. In another case, non-monogamous quantum correlation measures become monogamous, for moderately large number of parties [63].

The motivation behind this paper is three-fold. In this letter, we have asked (i) under what conditions monogamy property of quantum correlations is preserved, (ii) does monogamy for arbitrary pure multipartite state lead to monogamy of mixed states, and (iii) are there more general and stronger monogamy relations different from the standard one in Eq. (3). We prove that while a monogamous measure remains monogamous on raising its power, a non-monogamous measure remains non-monogamous on lowering its power. We also prove that monogamy of a convex quantum correlation measure for an arbitrary multipartite pure quantum state leads to its monogamy for the mixed state in the same Hilbert space. Monogamy of squared negativity for mixed states and that of entanglement of formation follow as direct corollaries. Authors of Ref. [80] have proposed following two conjectures regarding monogamy of squared entanglement of formation in multiparty systems: *the squared entanglement of formation may be monogamous for multipartite (i)* $2 \otimes d_2 \otimes \cdots \otimes d_n$, *and (ii) arbitrary d-dimensional, quantum systems*. Our previous result partially answers these conjectures in the sense that it now only remains to prove the monogamy of the squared entanglement of formation for pure states in arbitrary dimensions. We have further given hierarchical monogamy relations, and a strong monogamy inequality

$$\mathcal{Q}^\alpha(\rho_{AB}) \ge \frac{1}{2^{n-1}-1}\sum_X \mathcal{Q}^\alpha(\rho_{AX}) \ge \sum_j \mathcal{Q}^\alpha\left(\rho_{AB_j}\right), \tag{2}$$

where $X = \{B_{i_1}, \cdots, B_{i_k}\}$ is a nonempty proper subset of $B \equiv \{B_1, B_2, \cdots, B_n\}$, and $\alpha \ge 1$ is some positive real number.

This letter is organized as follows. In Section 2, we succinctly review the notion of monogamy of quantum correlations. While the main results of this letter are presented in Section 3, we give a summary in Section 4.

2. Monogamy of quantum correlations

Consider that $\mathcal{Q}$ is a bipartite correlation measure. If for a multipartite quantum system described by a state $\rho_{AB_1B_2\cdots B_n} \equiv \rho_{AB}$, the following inequality

$$\mathcal{Q}(\rho_{A(B_1\cdots B_n)}) \ge \sum_{j=1}^n \mathcal{Q}(\rho_{AB_j}), \tag{3}$$

holds, then the state ρ_{AB} is said to be monogamous under the quantum correlation measure $\mathcal{Q}$ [24,25]. Otherwise, it is non-monogamous. Moreover, the deficit between the two sides is referred to as monogamy score [81], and is given by

$$\delta\mathcal{Q} = \mathcal{Q}(\rho_{AB}) - \sum_{j=1}^n \mathcal{Q}(\rho_{AB_j}). \tag{4}$$

Monogamy score can be interpreted as residual entanglement of the bi-partition $A : rest$ of an n-party state that cannot be accounted for by the entanglement of two-qubit reduced density matrices separately.

It should be noted here that the monogamy inequality in Eq. (3) is just one constraint on the distribution of quantum correlations. Suppose $\mathcal{Q}$ does not obey the monogamy relation in Eq. (3), then is it non-monogamous? Can it be shared freely among the constituent parties? It may happen that it obeys the following constraint

$$\sum_{j=1}^n \mathcal{Q}(\rho_{AB_j}) \le b(\neq n), \tag{5}$$

and be still monogamous. Here we assume that $\mathcal{Q}$ is normalized, i.e., $0 \le \mathcal{Q} \le 1$. Numerical evidence of such a limitation was observed for entanglement of formation and concurrence in Ref. [72] for three-qubit systems.

Can there be more general and stronger monogamy relations than in Eq. (3)? Considerable attempts have been made to address this question from different perspectives [6,80,82–84] recently.

3. Results

In this section, we prove that a monogamous measure remains monogamous on raising its power, a non-monogamous measure remains non-monogamous on lowering its power, and monogamy of a convex quantum correlation measure for arbitrary multipartite pure quantum states leads to its monogamy for the mixed states. We also examine tighter monogamy inequalities compared to the standard one in Eq. (3), and hierarchical monogamy relations. Throughout our discussion we denote the multipartite quantum state $\rho_{AB_1B_2\cdots B_n}$ by ρ_{AB}, unless stated otherwise.