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Density waves in a system of non-interacting particles

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ABSTRACT

An ensemble of non-interacting bouncing balls being acted on by a constant gravitational force, starting at rest from a uniform density distribution, will develop a structure of sharply peaked density waves. We describe these waves by computing the density profile of such a system analytically, and we find that the analytical results are in good agreement with numerical findings. We suggest that in a real system, these density waves could be used to produce measurements of the strength of a gravitational field.

1. Introduction

Systems of elastically or inelastically bouncing balls have been a subject of interest in a number of different fields of physics. Researchers have used systems involving bouncing balls to study the behavior of dynamical systems, and especially the emergence and characteristics of chaos. Different variations of these systems are often complex enough to demonstrate physically interesting behavior while still being simple enough to be analytically tractable. One well-studied variant involves an oscillating or vibrating lower boundary with which the balls can exchange energy [1,2]. Such systems are often used as a model for Fermi acceleration [3–7]. A number of authors have studied the dynamical effects of collisions with differing degrees of elasticity or inelasticity [8,9]. Related systems have also been studied in the context of adding gravity to the classical billiard problem [10].

When used as a vehicle with which to better understand nonlinear dynamical systems, models of this kind can potentially be useful for understanding the behavior of a range of systems ranging from population biology to cryptography [11]. However, bouncing-ball models can also be useful in more direct ways. For instance, the complexities associated with bouncing dynamics may help to explain chaotic behavior in structures that contain pin joints with internal degrees of freedom [12].

Complex density wave structures can arise in systems without any inter-particle interactions, under the right conditions. One particularly simple example is a one-dimensional system of particles in a linear gravitational potential $V(x) = mgx$ with boundary conditions so that the particles bounce elastically at $x = 0$ (in this paper, the boundary will not vibrate or oscillate). If N particles begin at rest, uniformly distributed between $x = 0$ and $x = L$, we start to see a characteristic pattern of high-density pulses immediately. Specifically, we see a series of sawtooth-like structures that propagate from small x to large x before disappearing at $x = L$. As time passes, these saw-toothed waves become increasingly frequent and thinner. Fig. 1 shows a series of images of these saw-toothed patterns.

We will start by deriving an analytical expression for the density distribution of the particles in this system at an arbitrary time. We will observe in this expression the presence of saw-toothed wave structures that have all of the characteristics that we expect, and that are in good agreement with the results of numerical simulations, as shown in Fig. 2. We will also discuss possible applications of this phenomenon.

2. Partitioning parameter space

For a particle that starts at rest at a position x_0, it takes a total interval of time $\Delta t_f = \sqrt{2x_0/g}$ to fall to $x = 0$. Since the period of the particle's motion will be $2\Delta t_f$, it will be at $x = 0$ whenever $t = (2n + 1)\Delta t_f$ and back at $x = x_0$ for $t = 2n\Delta t_f$, where $n \in \{0, 1, 2, \ldots\}$.

At any particular time t, we can solve these expressions to find the set of particles currently at the maximal heights on their tra-

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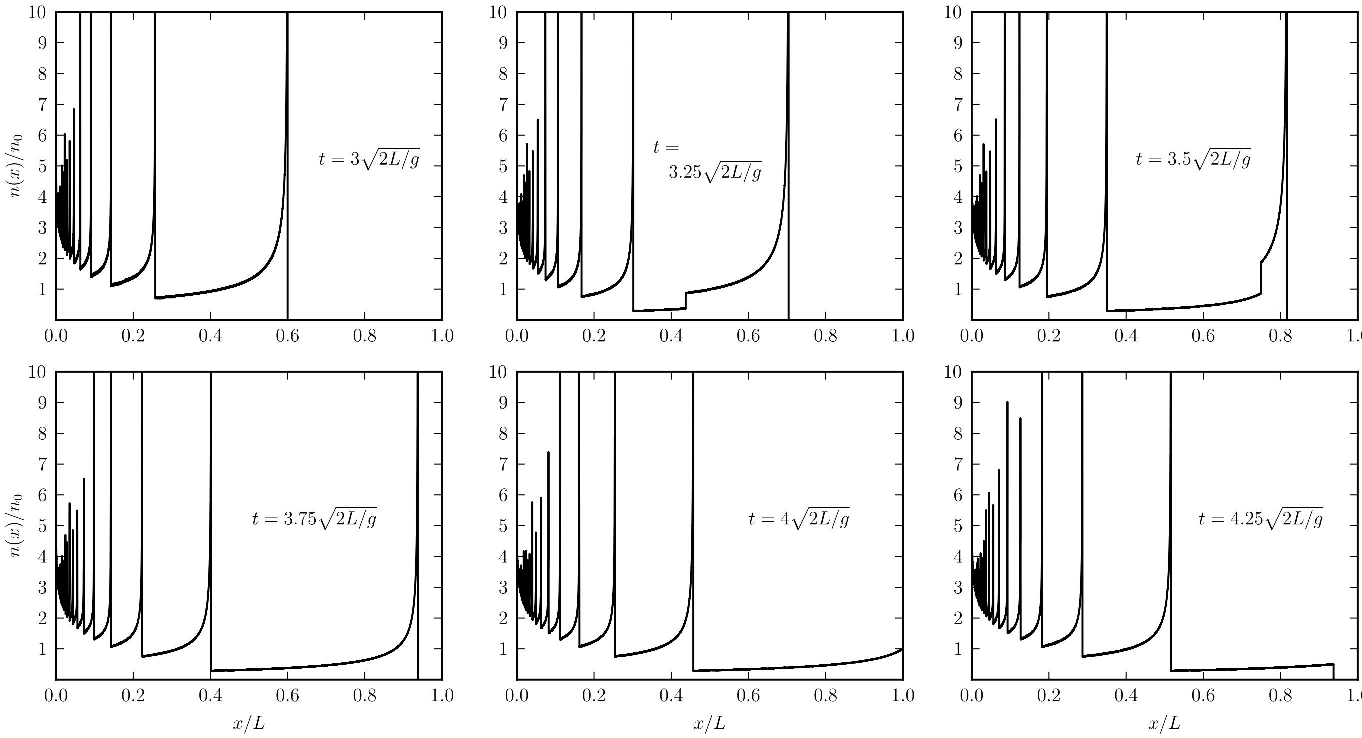

Fig. 1. These plots show snapshots of the density distribution with varying times. To get these distributions, we computed the positions of one million particles that started out uniformly distributed from 0 to L at $t = 0$.

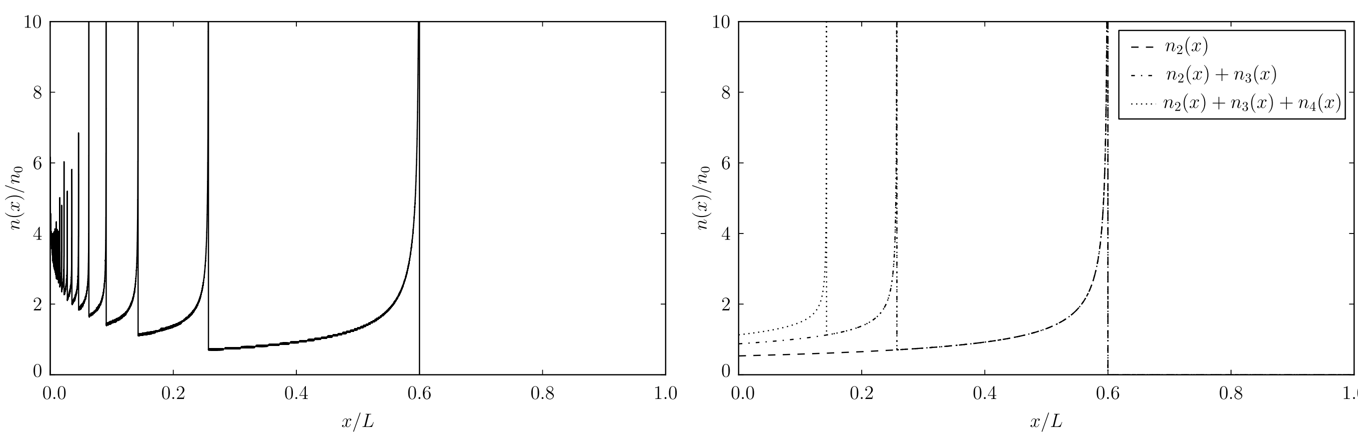

Fig. 2. The left plot shows a density distribution at $t = 3\sqrt{2L/g}$ obtained by computing the trajectories of one million particles. The right plot shows the density distributions corresponding to our analytical result for the first few nonempty R_k intervals.

jectories; these particles are defined by $x = x_0 = gt^2/8n^2$. A similar procedure gives the set of particles currently at $x = 0$ as $x_0 = gt^2/2(2n+1)^2$. Then the region in x_0-space of initial conditions for which particles will have bounced k times can be written as

$$R_k = \left\{ x_0 : \frac{gt^2}{2(2k+1)^2} \leq x_0 \leq \frac{gt^2}{2(2k-1)^2} \, , \, x_0 \leq L \right\} . \tag{1}$$

For any time $t > 0$, there will be regions that have bounced an arbitrarily large number of times, since the bounce period goes to zero as x_0 goes to zero. However, at later times, there will be a minimal k for which R_k is nonempty. We can find the minimal k by looking at the slowest-bouncing particle, which will have $x_0 = L$. The result is that

$$k_{\min} = \left\lfloor \frac{1}{2} + \frac{t}{2}\sqrt{\frac{g}{2L}} \right\rfloor . \tag{2}$$

3. From trajectories to density waves

Within each individual region R_k, we can solve for the constituent particles' trajectories explicitly. A particle that has bounced k times and that started at $x = x_0$ at $t = 0$ last bounced at $t_{\text{bounce}} = (2k-1)\sqrt{2x_0/g}$. Then until the particle leaves R_k (that is, until its next bounce), we can write its position as

$$x(t) = -\frac{g}{2}\,(t - t_{\text{bounce}})^2 + \sqrt{2gx_0}\,(t - t_{\text{bounce}}) . \tag{3}$$

From this expression, we would like to get a density distribution $n_k(x)$ for the elements of R_k, in the continuous limit (so that we imagine initially having n_0 particles per unit length). Now, in the limit as $k \to \infty$, this is an easier problem: the region R_k becomes arbitrarily thin in x_0-space, so the particles in R_k just trace out the trajectory of the kth bounce of a particle that starts at x_0. In this case, the origin of a saw-toothed density distribution is intuitively clear, since we can just invert the velocity of the particle to get