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The tri-Hamiltonian dual system of supersymmetric two boson system

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ABSTRACT

The dual system of the supersymmetric two boson system is constructed through the approach of tri-Hamiltonian duality, and inferred from this duality, its zero-curvature representation is also figured out. Furthermore, the dual system is shown to be equivalent to a $N = 2$ supersymmetric Camassa–Holm equation, and this relation results in a new linear spectral problem for the $N = 2$ supersymmetric Camassa–Holm equation.

1. Introduction

As main contents of modern theory of integrable systems, soliton equations exhibit extremely rich mathematical properties and wide physical applications. In half a century, classical soliton equations [1], namely, partial differential equations sharing common features with the famous Korteweg–de Vries (KdV) equation [2], were generalized to various cases, such as discrete [3], non-commutative [4], super/supersymmetric etc., which extensively widen the scope of integrability. This paper is concerned with supersymmetric integrable equations.

During the past four decades, a number of classical integrable systems have been embedded in their supersymmetric counterparts, and as results the supersymmetric sine-Gordon equation [5], the supersymmetric KdV equation [6] and the supersymmetric nonlinear Schrödinger equation [7] have been constructed. Besides these prototypes, some classical non-evolutionary integrable equations, like the Hunter–Saxton equation [8] and the Camassa–Holm equation [9], were also successfully generalized to superspaces through different approaches. For instance, even and odd supersymmetric Hunter–Saxton equations were derived out as negative flows of supersymmetric Harry Dym hierarchies [10,11], and the even supersymmetric Hunter–Saxton equation was also shown to describe the geodesic flow on the space of superdiffeomorphisms of the circle that leaves a point fixed endowed with a right-invariant metric [12]. Some unequivalent supersymmetric Camassa–Holm equations were constructed via deformation of $N = 2$ superconformal algebra [13] or geodesic equations on superconformal group [14], but as far as we know, their integrability is still not clear.

The method of tri-Hamiltonian duality, developed by Fokas, Fuchssteiner, Olver and Rosenau, is an effective tool to generate new bi-Hamiltonian systems from known ones, and produces many integrable equations of Camassa–Holm type. Indeed, it is Fuchssteiner [15] who first presented the Camassa–Holm equation (up to a misprint) by means of recombining the Hamiltonian operators of the KdV equation. This method traces back to Fokas and Fuchssteiner [16] and has been elaborated [17,18]. In particular, Olver and Rosenau made further contribution to this method and various examples were considered [19]. New systems constructed in this way are termed as dual systems of the original ones. In the same year, Schiff constructed the zero-curvature representations for those dual equations [20]. The method of tri-Hamiltonian duality was adopted by Popowicz to $N = 2$ supersymmetric KdV equations, and generated $N = 2$ supersymmetric Camassa–Holm equations [21] admitting bi-Hamiltonian structures.

In this paper, we manage to construct the dual system of a supersymmetric bi-Hamiltonian system, which is equivalent to the supersymmetric two boson (sTB) system, and present a zero-curvature representation for this dual system in the scheme of tri-Hamiltonian duality. The sTB system, proposed by Brunelli and Das [22], serves as a supersymmetric extension of the dispersive water wave equation [23,24], also referred to as Kaup–Borer or classical Boussinesq system in literatures, and is formulated as

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$$\begin{cases} \phi_{0,t} = -\phi_{0,xx} + (\mathcal{D}(\mathcal{D}\phi_0)^2) + 2\phi_{1,x}, \\ \phi_{1,t} = \phi_{1,xx} + 2((\mathcal{D}\phi_0)\phi_1)_x, \end{cases}$$

where $\phi_i = \phi_i(x, \theta, t)(i = 0, 1)$ are fermionic (Grassmann odd) superfields depending on superspatial variables (x, θ) and temporal variable t, and $\mathcal{D}$ denotes the superderivative defined as $\mathcal{D} = \partial_\theta + \theta\partial_x$. Furthermore, Brunelli and Das have shown that the sTB system is a bi-Hamiltonian system

$$\begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix}_t = \mathcal{K}_1 \begin{pmatrix} \frac{\delta H_3}{\delta \phi_0} \\ \frac{\delta H_3}{\delta \phi_1} \end{pmatrix} = \mathcal{K}_2 \begin{pmatrix} \frac{\delta H_2}{\delta \phi_0} \\ \frac{\delta H_2}{\delta \phi_1} \end{pmatrix},$$

where both Hamiltonian operators are

$$\mathcal{K}_1 = \begin{pmatrix} 0 & -\mathcal{D} \\ -\mathcal{D} & 0 \end{pmatrix}, \quad \mathcal{K}_2 = \begin{pmatrix} -2\mathcal{D} - 2\mathcal{D}^{-1}\phi_1\mathcal{D}^{-1} + \mathcal{D}^{-1}(\partial\phi_0)\mathcal{D}^{-1} & \mathcal{D}^3 - \mathcal{D}(\mathcal{D}\phi_0) + \mathcal{D}^{-1}\phi_1\mathcal{D} \\ -\mathcal{D}^3 - (\mathcal{D}\phi_0)\mathcal{D} - \mathcal{D}\phi_1\mathcal{D}^{-1} & -\phi_1\partial - \partial\phi_1 \end{pmatrix},$$

and corresponding Hamiltonians are given by

$$H_2 = -\int (\mathcal{D}\phi_0)\phi_1 \mathrm{d}x\mathrm{d}\theta, \quad H_3 = \int \Big((\mathcal{D}\phi_{0,x}) - (\mathcal{D}\phi_0)^2 - (\mathcal{D}\phi_1) \Big)\phi_1 \mathrm{d}x\mathrm{d}\theta.$$

Interestingly as stated by Liu [25], the sTB system could be reformulated with two local Hamiltonian operators by introducing new variables $(u, \chi) \equiv (-(\mathcal{D}\phi_0), \phi_1)$. Under this change of variables, the sTB equation is transformed to

$$\begin{cases} u_t = -u_{xx} - 2uu_x - 2(\mathcal{D}\chi)_x, \\ \chi_t = \chi_{xx} - 2(u\chi)_x, \end{cases} \tag{1}$$

and Hamiltonian operators $\mathcal{K}_1$ and $\mathcal{K}_2$ are respectively converted to

$$\mathcal{B}_1 = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}, \ \mathcal{B}_2 = \begin{pmatrix} -2\mathcal{D}^3 - 2\chi - (\mathcal{D}u) & -\partial^2 - \partial u - \chi\mathcal{D} \\ \partial^2 - u\partial + \mathcal{D}\chi & -\chi\partial - \partial\chi \end{pmatrix},$$

where the superfield u is bosonic, while the other superfield χ is fermionic. It is worth noting that both Hamiltonian operators $\mathcal{B}_1$ and $\mathcal{B}_2$ are local. The system (1) will be also referred to as the sTB system hereinafter, and its dual system and zero-curvature representation will be established.

The paper is organized as follows. In section 2, a new pair of compatible Hamiltonian operators is presented by rearranging $\mathcal{B}_1$ and $\mathcal{B}_2$, and generates the dual system of the sTB system (1). Inferred from this duality, we formulate a zero-curvature representation for the dual sTB system in section 3. Section 4 is devoted to explain the relation between the dual sTB system and a $N = 2$ supersymmetric Camassa–Holm equation, which first appeared as a dual system of the SKdV$_4$ equation [21], and later was rediscovered by Lenells and Lechtenfeld as the Euler equation on superconformal algebra [26]. Conclusions will be given in the last section.

2. The dual sTB system

Decomposing the Hamiltonian operator $\mathcal{B}_2$ by extracting leading terms, we obtain

$$\mathcal{B}_1 = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}, \quad \mathcal{B}_2^{(1)} = \begin{pmatrix} -2\mathcal{D}^3 & -\partial^2 \\ \partial^2 & 0 \end{pmatrix}, \quad \mathcal{B}_2^{(2)} = \begin{pmatrix} -2\chi - (\mathcal{D}u) & -\partial u - \chi\mathcal{D} \\ -u\partial + \mathcal{D}\chi & -\chi\partial - \partial\chi \end{pmatrix}.$$

About the triplet $\mathcal{B}_1$, $\mathcal{B}_2^{(1)}$ and $\mathcal{B}_2^{(2)}$, we have

Proposition 1. *For arbitrary constants a, b and c, the skew-symmetric operator*

$$\mathcal{E} = a\mathcal{B}_1 + b\mathcal{B}_2^{(1)} + c\mathcal{B}_2^{(2)}$$

is a Hamiltonian operator.

Proof. Since the operator $\mathcal{E}$ is obviously skew-symmetric, one just need to show that the Poisson bracket defined by $\mathcal{E}$ satisfies the super Jacobi identity, or equivalently to show that

$$\langle \alpha, \mathcal{E}'[\mathcal{E}\beta]\gamma \rangle + \langle \beta, \mathcal{E}'[\mathcal{E}\gamma]\alpha \rangle + \langle \gamma, \mathcal{E}'[\mathcal{E}\alpha]\beta \rangle = 0$$

for arbitrary 2-dimensional testing vectors α, β and γ.

Because all calculations are straightforward, details are omitted. □

Various corollaries may be inferred from Proposition 1. On the one hand, when $b = c$ it manifests the compatibilities of Hamiltonian operators $\mathcal{B}_1$ and $\mathcal{B}_2$. On the other hand, if $a = b$, then we have a new pair of compatible Hamiltonian operators, i.e.

$$\hat{\mathcal{B}}_1 = \mathcal{B}_1 + \mathcal{B}_2^{(1)} \quad \text{and} \quad \hat{\mathcal{B}}_2 = \mathcal{B}_2^{(2)},$$

on which the dual system of the sTB equation (1) would be established.