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Functional integral derivation of the kinetic equation of two-dimensional point vortices

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ABSTRACT

We present a brief derivation of the kinetic equation describing the secular evolution of point vortices in two-dimensional hydrodynamics, by relying on a functional integral formalism. We start from Liouville's equation which describes the exact dynamics of a two-dimensional system of point vortices. At the order $1/N$, the evolution of the system is characterised by the first two equations of the BBGKY hierarchy involving the system's 1-body distribution function and its 2-body correlation function. Thanks to the introduction of auxiliary fields, these two evolution constraints may be rewritten as a functional integral. When functionally integrated over the 2-body correlation function, this rewriting leads to a new constraint coupling the 1-body distribution function and the two auxiliary fields. Once inverted, this constraint provides, through a new route, the closed non-linear kinetic equation satisfied by the 1-body distribution function. Such a method sheds new lights on the origin of these kinetic equations complementing the traditional derivation methods.

1. Introduction

There exist beautiful analogies between stellar systems and two-dimensional ($2D$) vortices [1]. Stellar systems and $2D$ point vortices undergo two successive types of relaxation. They first reach a quasistationary state (QSS) due to a process of violent collisionless relaxation. The concept of violent relaxation was introduced by Lynden-Bell [2] in the case of stellar systems described by the Vlasov equation and by Miller [3] and Robert and Sommeria [4] in the case of $2D$ vortices described by the $2D$ Euler equation (see [5] for a description of the close link between these two theories). These QSSs correspond to galaxies in astrophysics [6] or to large scale vortices (like Jupiter's great red spot) in geophysical and astrophysical flows [7]. On a longer (secular) timescale, "collisions"[1] between stars or between point vortices come into play and drive the system towards a statistical equilibrium state described by the Boltzmann distribution. This statistical equilibrium state was conjectured by Ogorodnikov [8] in the case of stellar systems and by Onsager [9,10] and Montgomery and Joyce [11] in the case of $2D$ point vortices. Actually, for collisional stellar systems such as globular clusters the relaxation towards the Boltzmann statistical equilibrium state is hampered by the evaporation of stars [12] and by the gravothermal catastrophe [13,14]. In the case of $2D$ point vortices, the statistical equilibrium state may present the peculiarity to have a negative temperature as first noted by Onsager [9].

To understand the dynamical evolution of these systems, we need to develop a kinetic theory. The collisionless evolution of stellar systems is described by the Vlasov [15] equation that was first written by Jeans [16] in astrophysics.[2] The collisional evolution of stellar systems is usually described by the Fokker–Planck equation introduced by Chandrasekhar [17] or by the Landau [18] equation. These equations rely on a local approximation (as if the system were spatially homogeneous) and neglect collective effects (i.e., the dressing of the stars by their polarisation cloud). A gravitational Landau equation that takes into account spatial inhomogeneity through the use of angle-action variables has been introduced in [19–21] and a gravitational Balescu–Lenard equation that takes into account spatial inhomogeneity and collective effects

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[1] These "collisions" do not correspond to physical collisions but rather to – possibly distant – encounters between the particles. They account for fluctuations due to finite-N effects, i.e., for the granularity of the system.

[2] The kinetic theories of stellar systems and neutral Coulombian plasmas have been developed in parallel (and often independently) by astrophysicists and plasma physicists.

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has been introduced in [22,23]. These equations have recently been applied to stellar discs in [24–26].

Exploiting the analogy between $2D$ vortices and stellar systems, a kinetic theory of point vortices has been elaborated by Chavanis [27]. The collisionless evolution of point vortices is described by the $2D$ Euler equation. When collective effects are neglected, the collisional evolution of point vortices is described by a Landau-type equation [27–29]. A Balescu–Lenard-type equation taking collective effects into account has been derived in [30,31] for an axisymmetric distribution of point vortices. It is equivalent to the one derived in [32] in the similar context of non-neutral plasmas.

One can understand the collisional evolution of stellar systems and $2D$ point vortices heuristically by analogy with the Brownian motion. A star has a diffusive motion due to the fluctuations of the gravitational force but it also experiences a dynamical friction [33]. Similarly, a point vortex has a diffusive motion due to the fluctuations of the velocity field and also experiences a systematic drift [27]. The diffusion can be understood by considering the statistics of the gravitational force created by a random distribution of stars [34] or the statistics of the velocity created by a random distribution of point vortices [35]. The dynamical friction experienced by a star and the systematic drift experienced by a point vortex can be understood from a polarisation process and a linear response theory (see [36] for stellar systems and [37] for point vortices). The friction and drift coefficients are related to the diffusion coefficient by a form of Einstein relation. Further analogies between the kinetic theory of stellar systems, $2D$ vortices, and systems with long-range interactions in general are discussed in [20].

There are many methods to derive kinetic equations for systems with long-range interactions. The most popular are the BBGKY hierarchy based on the Liouville equation (see [38,39] for plasmas, [40–42,22,21] for stellar systems and [29,31] for point vortices), the quasilinear theory based on the Klimontovich equation (see [43] for plasmas, [44,23] for stellar systems and [32,30,29] for point vortices), and the projection operator technique also based on the Liouville equation (see [45] for stellar systems and [27] for point vortices). One can also derive kinetic equations from a field theory. This method was introduced by Jolicoeur and Le Guillou [46] to derive the homogeneous Balescu-Lenard equation of plasma physics. Recently, this method was generalised to stellar systems in [47] to derive the inhomogeneous Landau equation. Owing to the analogy between stellar systems and $2D$ point vortices, it is of interest to show how this method can be used to derive the Landau equation for axisymmetric point vortices.

The present letter is organised as follows. Section 2 presents a brief derivation of the relevant BBGKY hierarchy in the context of the kinetic theory of $2D$ point vortices. Section 3 details the functional integral formalism introduced in [46] and applied in [47] for inhomogeneous long-range systems. Section 4 illustrates how this formalism may be used to obtain the Landau equation describing the secular evolution of axisymmetric $2D$ point vortices. Section 5 discusses the limitations of our approach and its possible extensions. Finally, section 6 wraps up.

2. Derivation of the BBGKY hierarchy

In this section, we briefly recover the evolution equations describing the dynamics of point vortices and the associated BBGKY hierarchy. We consider a $2D$ system made of N point vortices of individual circulation $\gamma = \Gamma_{\mathrm{tot}}/N$. The individual dynamics of these vortices is entirely described by the Kirchhoff–Hamilton equations which read [48]:

$$\gamma \frac{\mathrm{d}x_i}{\mathrm{d}t} = \frac{\partial H}{\partial y_i} \;\; ; \;\; \gamma \frac{\mathrm{d}y_i}{\mathrm{d}t} = -\frac{\partial H}{\partial x_i} , \tag{1}$$

where we introduced the coordinates $\boldsymbol{r} = (x, y)$, as well as the Hamiltonian $H = \gamma^2 \sum_{i<j} u_{ij}$, where $u_{ij} = -1/(2\pi)\ln(|\boldsymbol{r}_i - \boldsymbol{r}_j|)$ is the potential of interaction between two vortices. We may now introduce the N-body probability distribution function (PDF) $P_N(\boldsymbol{r}_1, ..., \boldsymbol{r}_N, t)$, which describes the probability of finding the vortex 1 at position $\boldsymbol{r}_1$, vortex 2 at position $\boldsymbol{r}_2$, etc. We normalise P_N such that $\int \mathrm{d}\boldsymbol{r}_1 ... \mathrm{d}\boldsymbol{r}_N \, P_N(\boldsymbol{r}_1, ..., \boldsymbol{r}_N, t) = 1$. The evolution of P_N is then governed by Liouville's equation which reads

$$\frac{\partial P_N}{\partial t} + \gamma \sum_{i=1}^{N} \boldsymbol{V}_i \cdot \frac{\partial P_N}{\partial \boldsymbol{r}_i} = 0 , \tag{2}$$

where we defined the velocity $\boldsymbol{V}_i = \sum_{j \neq i} \boldsymbol{V}_{ij} = \sum_{j \neq i} -\boldsymbol{e}_z \times \partial u_{ij}/\partial \boldsymbol{r}_i$. Here, $\boldsymbol{V}_{ij}$ denotes the exact velocity induced by the vortex j on the vortex i. We now introduce the reduced distribution functions (DF) f_n as

$$f_n(\boldsymbol{r}_1, ..., \boldsymbol{r}_n, t) = \gamma^n \frac{N!}{(N-n)!} \int \mathrm{d}\boldsymbol{r}_{n+1} ... \mathrm{d}\boldsymbol{r}_N \, P_N(\boldsymbol{r}_1, ..., \boldsymbol{r}_N, t) . \tag{3}$$

Integrating equation (2) w.r.t. $(\boldsymbol{r}_{n+1}, ..., \boldsymbol{r}_N)$, one obtains a BBGKY-like hierarchy of equations as

$$\frac{\partial f_n}{\partial t} + \sum_{i=1}^{n} \sum_{k=1, k\neq i}^{n} \gamma \boldsymbol{V}_{ik} \cdot \frac{\partial f_n}{\partial \boldsymbol{r}_i} + \sum_{i=1}^{n} \int \mathrm{d}\boldsymbol{r}_{n+1} \, \boldsymbol{V}_{i,n+1} \cdot \frac{\partial f_{n+1}}{\partial \boldsymbol{r}_i} = 0 . \tag{4}$$

We are interested in the contributions arising from the correlations between particles, and therefore introduce the cluster representation of the DF. Indeed, we define the 2- and 3-body correlation functions g_2 and g_3 as

$$\begin{aligned} f_2(\boldsymbol{r}_1, \boldsymbol{r}_2) &= f_1(\boldsymbol{r}_1) f_1(\boldsymbol{r}_2) + g_2(\boldsymbol{r}_1, \boldsymbol{r}_2) , \\ f_3(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3) &= f_1(\boldsymbol{r}_1) f_1(\boldsymbol{r}_2) f_1(\boldsymbol{r}_3) \\ &\quad + f_1(\boldsymbol{r}_1) g_2(\boldsymbol{r}_2, \boldsymbol{r}_3) + f_1(\boldsymbol{r}_2) g_2(\boldsymbol{r}_1, \boldsymbol{r}_3) \\ &\quad + f_1(\boldsymbol{r}_3) g_2(\boldsymbol{r}_1, \boldsymbol{r}_2) + g_3(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3) . \end{aligned} \tag{5}$$

It is then straightforward to check that one has the normalisations

$$\begin{aligned} &\int \mathrm{d}\boldsymbol{r}_1 \, f_1(\boldsymbol{r}_1) = \gamma N \;\; ; \;\; \int \mathrm{d}\boldsymbol{r}_1 \mathrm{d}\boldsymbol{r}_2 \, g_2(\boldsymbol{r}_1, \boldsymbol{r}_2) = -\gamma^2 N , \\ &\int \mathrm{d}\boldsymbol{r}_1 \mathrm{d}\boldsymbol{r}_2 \mathrm{d}\boldsymbol{r}_3 \, g_3(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3) = 2\gamma^3 N . \end{aligned} \tag{6}$$

Since the individual circulation scales like $\gamma \sim 1/N$, one immediately has $|f_1| \sim 1$, $|g_2| \sim 1/N$, and $|g_3| \sim 1/N^2$. In order to consider quantities of order 1, we introduce the system's 1-body DF F, and 2-body correlation function $\mathcal{C}$ as

$$F = f_1 \;\; ; \;\; \mathcal{C} = \frac{g_2}{\gamma} . \tag{7}$$

When truncated at the order $1/N$, one can easily show that the first two equations of the hierarchy from equation (4) become

$$\frac{\partial F}{\partial t} + \left[\int \mathrm{d}\boldsymbol{r}_2 \, \boldsymbol{V}_{12} F(\boldsymbol{r}_2) \right] \cdot \frac{\partial F}{\partial \boldsymbol{r}_1} + \gamma \int \mathrm{d}\boldsymbol{r}_2 \, \boldsymbol{V}_{12} \cdot \frac{\partial \mathcal{C}(\boldsymbol{r}_1, \boldsymbol{r}_2)}{\partial \boldsymbol{r}_1} = 0 , \tag{8}$$

and

$$\begin{aligned} &\frac{1}{2} \frac{\partial \mathcal{C}(\boldsymbol{r}_1, \boldsymbol{r}_2)}{\partial t} + \left[\int \mathrm{d}\boldsymbol{r}_3 \, \boldsymbol{V}_{13} F(\boldsymbol{r}_3) \right] \cdot \frac{\partial \mathcal{C}(\boldsymbol{r}_1, \boldsymbol{r}_2)}{\partial \boldsymbol{r}_1} \\ &\quad + \boldsymbol{V}_{12} \cdot \frac{\partial F}{\partial \boldsymbol{r}_1} F(\boldsymbol{r}_2) + \left[\int \mathrm{d}\boldsymbol{r}_3 \, \boldsymbol{V}_{13} \mathcal{C}(\boldsymbol{r}_2, \boldsymbol{r}_3) \right] \cdot \frac{\partial F}{\partial \boldsymbol{r}_1} \\ &\quad + (1 \leftrightarrow 2) = 0 , \end{aligned} \tag{9}$$