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Hybrid resonance and long-time asymptotic of the solution to Maxwell's equations [☆]

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ABSTRACT

We study the long-time asymptotic of the solutions to Maxwell's equation in the case of an upper-hybrid resonance in the cold plasma model. We base our analysis in the transfer to the time domain of the recent results of B. Després, L.M. Imbert-Gérard and R. Weder (2014) [15], where the singular solutions to Maxwell's equations in the frequency domain were constructed by means of a limiting absorption principle and a formula for the heating of the plasma in the limit of vanishing collision frequency was obtained. Currently there is considerable interest in these problems, in particular, because upper-hybrid resonances are a possible scenario for the heating of plasmas, and since they can be a model for the diagnostics involving wave scattering in plasmas.

1. Introduction

In plasma physics [5,7,17] upper-hybrid resonances may develop at places where a density gradient of charged particles excited by a strong background magnetic field generates singular solutions to Maxwell's equations. This phenomenon shows up in propagation of electromagnetic waves in the outer region of the atmosphere, as explained first in [6]. It also appears in reflectometry experiments [21,11] and in heating devices in fusion plasma [16] in Tokamaks. An important feature in this direction is the energy deposit which is finite. It may exceed by far the energy exchange which occurs in Landau damping [17,25]. Notice however that there exist situations where hybrid resonance and Landau damping are modeled in a unique set [13,26]. Furthermore, our model could be applied in diagnostics involving wave scattering at the upper-hybrid resonance [21,10].

The starting point of the analysis is the linearized Vlasov–Maxwell's equations of a non-homogeneous plasma around a bulk magnetic field $\mathbf{B}_0 \neq 0$. It yields the non-stationary Maxwell's equations with a linear current

$$\begin{cases} -\frac{1}{c^2}\partial_t \mathbf{E} + \nabla \wedge \mathbf{B} = \mu_0 \mathbf{J}, \qquad \mathbf{J} = -eN_e \mathbf{u}_e, \\ \partial_t \mathbf{B} + \nabla \wedge \mathbf{E} = 0, \\ m_e \partial_t \mathbf{u}_e = -e(\mathbf{E} + \mathbf{u}_e \wedge \mathbf{B}_0) - m_e \nu \mathbf{u}_e. \end{cases} \tag{1.1}$$

The electric field is $\mathbf{E}$ and the magnetic field is $\mathbf{B}$. The modulus of the background magnetic field $|\mathbf{B}_0|$ and its direction $\mathbf{b}_0 = \frac{\mathbf{B}_0}{|\mathbf{B}_0|}$ will be assumed constant in space for simplicity in our work. The total magnetic field is expanded as first order as $\mathbf{B}_{tot} = \mathbf{B}_0 + \mathbf{B}$. Note that in the last equation in (1.1) $\mathbf{B}$ is neglected. The absolute value of the charge of electrons is e, the mass of electrons is m_e, the velocity of light is $c = \sqrt{\frac{1}{\varepsilon_0 \mu_0}}$ where the permittivity of vacuum is ε_0 and the permeability of vacuum is μ_0. The third equation corresponds to moving electrons with velocity $\mathbf{u}_e$, and the electronic density N_e is a given function of the space variables. One assumes the existence of a bath of particles which is the reason of the friction between the electrons and the bath of particles with collision frequency ν. Much more material about such models can be found in classical physical textbooks [5,17]. The loss of energy in domain Ω can easily be computed in the time domain starting from (1.1). One obtains

$$\frac{d}{dt}\int_\Omega \left(\frac{\varepsilon_0 |\mathbf{E}|^2}{2} + \frac{|\mathbf{B}|^2}{2\mu_0} + \frac{m_e N_e |\mathbf{u}_e|^2}{2} \right) = -\int_\Omega \nu m_e N_e |\mathbf{u}_e|^2 + \text{boundary terms}.$$

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Therefore,

$$\mathcal{Q}(\nu) = \int_{\Omega} \nu m_e N_e |\mathbf{u}_e|^2 \quad (1.2)$$

represents the total loss of energy of the electromagnetic field plus the electrons in function of the collision frequency ν. Since the energy loss is necessarily equal to what is gained by the bath of particles, it will be referred to as the heating. In fusion plasma, a value of $\nu \approx 10^{-7}$ in relative units is often encountered. It is therefore tempting to set the friction parameter, i.e. the collision frequency ν, equal to zero, but this naive approach is incorrect, as we explain below.

Equations (1.1) can be written in the frequency domain, where ω is the frequency, that is $\partial_t = -i\omega$ where for simplicity of the notations $\omega > 0$. We assume that the bulk magnetic field is along the z coordinate, i.e. $\mathbf{b}_0 = (0, 0, 1)$. We obtain,

$$\begin{cases} \frac{1}{c^2} i\omega \mathbf{E} + \nabla \wedge \mathbf{B} = -\mu_0 e N_e \mathbf{u}_e, \\ -i\omega \mathbf{B} + \nabla \wedge \mathbf{E} = 0, \\ -i m_e \omega \mathbf{u}_e = -e(\mathbf{E} + \mathbf{u}_e \wedge \mathbf{B}_0) - m_e \nu \mathbf{u}_e. \end{cases} \quad (1.3)$$

One can compute the velocity using the third equation $\widetilde{\omega}\mathbf{u}_e + \omega_c i \mathbf{u}_e \wedge \mathbf{b}_0 = -\frac{e}{m_e} i\mathbf{E}$ where the cyclotron frequency is $\omega_c = \dfrac{e|\mathbf{B}_0|}{m_e}$. The frequency $\widetilde{\omega} = \omega + i\nu$ is shifted in the complex plane by a factor equal to the friction parameter. It is then easy to eliminate $\mathbf{u}_e$ from the first equation of the system (1.3) and to obtain the time harmonic Maxwell's equation

$$\nabla \wedge \nabla \wedge \mathbf{E} - \left(\frac{\omega}{c}\right)^2 \underline{\underline{\varepsilon}}(\nu)\mathbf{E} = 0. \quad (1.4)$$

The dielectric tensor is the one of the cold plasma approximation, the so-called Stix tensor, [7,17]

$$\underline{\underline{\varepsilon}}(\nu) = \begin{pmatrix} 1 - \frac{\widetilde{\omega}\omega_p^2}{\omega(\widetilde{\omega}^2 - \omega_c^2)} & i\frac{\omega_c \omega_p^2}{\omega(\widetilde{\omega}^2 - \omega_c^2)} & 0 \\ -i\frac{\omega_c \omega_p^2}{\omega(\widetilde{\omega}^2 - \omega_c^2)} & 1 - \frac{\widetilde{\omega}\omega_p^2}{\omega(\widetilde{\omega}^2 - \omega_c^2)} & 0 \\ 0 & 0 & 1 - \frac{\omega_p^2}{\omega\widetilde{\omega}} \end{pmatrix}. \quad (1.5)$$

The parameters of the dielectric tensor are the cyclotron frequency ω_c and the plasma frequency $\omega_p = \sqrt{\frac{e^2 N_e}{\varepsilon_0 m_e}}$ which depends on the electronic density N_e. We are interested in the physical situation where the electronic density N_e is not constant, that is $\nabla N_e \neq 0$. Observe that the cyclotron frequency $\omega = \omega_c$ is a singularity of the dielectric tensor. In this paper we consider $\omega \neq \omega_c$, hence, the dielectric tensor (1.5) is smooth. In fact, we focus on the paradoxical upper-hybrid resonance that appears when $\omega = \omega_h := \sqrt{\omega_c^2 + \omega_p^2}$. In plasma physics ω_h is called the upper-hybrid frequency.

If we set $\nu = 0$ the first two diagonal entries in the Stix tensor are equal to $\frac{\omega^2 - \omega_H^2}{\omega^2 - \omega_p^2}$. The crucial issue is that they are equal to zero when $\omega = \omega_H$ and that they continuously change in sign when ω increases from values smaller than ω_H to values bigger than ω_H. For this reason (see [15]) the system (1.4) with $\nu = 0$ is ill posed: the solution is not unique and, furthermore, it has singular solutions that contain distributions. The way out of this dilemma [15] is a limiting absorption principle, we take ν to zero in a limiting sense: We consider $\nu > 0$ small and we construct an unique solution to (1.4) characterized by its behavior at the point in space where $\omega_H = \omega$ and by demanding that it goes to zero at spacial infinity, away from the sources of the electromagnetic field (the system (1.3) is assumed to be coercive (non-propagating) at infinity). We call it the singular solution. We then prove that as $\nu \downarrow 0$ the singular solution converges (in distribution sense) to a limiting singular solution that is the appropriate physical solution to our problem. Furthermore, we give a formula for the limiting heating $\mathcal{Q}(0^+) := \lim_{\nu \downarrow 0} \mathcal{Q}(\nu)$, that turns out to be positive. The fact $Q(0^+) > 0$ implies that the hybrid resonance is able to transfer a finite amount of energy from the electromagnetic field and the electrons to the bath of particles (i.e. to heat bath of particles) even in the limit when $\nu = 0$. This is, indeed, a remarkable result. The physical interpretation is that as ν goes to zero – and the solution becomes singular – the velocity of the electrons increases (since the friction with the ions goes to zero) and there is a compensation in the right-hand side of (1.2). In the end, the increase in the velocity of the electrons dominates, so that, in the limit the heating $\mathcal{Q}(0^+)$ is positive. Moreover, from the mathematical point of view, the fact that the singular solution with $\nu > 0$ converges to a limiting singular solution implies that for ν small the singular solution is close to the limiting singular solution and, hence, it does not change very much with ν. This means that a small ν positive can be used as a regularization parameter to numerically compute the limiting singular solution and the limiting heating $\mathcal{Q}(0^+)$. This numerical scheme has been successfully used in the Ph.D. thesis [20] that contains extensive numerical calculations. It was found that the numerical solution with small $\nu > 0$ converges fast to the exact solution (with $\nu = 0$) in point-wise sense, except of course, at the singularity. Moreover, a large fraction of the energy of the incoming wave may be absorbed by the bath of particles, up to 95% in the case of normal incidence, and up to 76.7% in the case of oblique incidence. Our results in [15,20], in particular our formula for the heating $\mathcal{Q}(0^+)$ shows, in a rigorous and quantitative way, that upper-hybrid resonances are, indeed, an efficient method to heat the bath of particles.

We now present, for later use, our results in [15] in a precise way. To study the upper-hybrid resonance we consider the 2×2 upper-left block in (1.4), that corresponds to the transverse electric (TE) mode, $E = (E_x, E_y, 0)$, where the electric field is transverse to the bulk magnetic field $\mathbf{B}_0$. We assume that we have a slab geometry: all coefficients in (1.5) depend only on the coordinate x. Furthermore, we suppose that E_x, E_y, are independent of z, that is the coordinate along the bulk magnetic field $\mathbf{B}_0$.

Then, in the limit case $\nu = 0$, the 2×2 upper-left block in (1.4) gives

$$\begin{cases} W + \partial_y E_x - \partial_x E_y = 0, \\ \partial_y W - \alpha E_x - i\delta E_y = 0, \\ -\partial_x W + i\delta E_x - \alpha E_y = 0, \end{cases} \quad (1.6)$$

where we find convenient to introduce the vorticity $W := \partial_x E_y - \partial_y E_x$ that is proportional to the magnetic field B_z. The coefficients α, δ are equal to

$$\alpha = \frac{\omega^2}{c^2}\left(1 - \frac{\omega_p^2}{\omega^2 - \omega_c^2}\right) \quad \text{and} \quad \delta = \frac{\omega^2}{c^2} \times \frac{\omega_c \omega_p^2}{\omega(\omega^2 - \omega_c^2)}. \quad (1.7)$$

In plasma physics the system (1.6) is called the equations for the X-mode, or the extraordinary mode, and also the extraordinary wave.

We consider the system (1.6) in the two dimensional domain,

$$\Omega = \left\{(x, y) \in \mathbb{R}^2, \quad -L \le x, \quad y \in \mathbb{R}, \quad L > 0\right\}.$$

We assume the non-homogeneous boundary condition

$$W + i\lambda n_x E_y = g \text{ on the left boundary } x = -L, \qquad \lambda > 0, \quad (1.8)$$

that corresponds to a source, like a radiation antenna that is used to heat the plasma. We suppose that α and δ satisfy conditions that correspond to an upper-hybrid resonance at $x = 0$. The main assumptions are: The function α is twice continuous differentiable and δ is continuous with continuous first derivative. Furthermore, $\alpha(0) = 0, \alpha'(0) < 0$, and $\alpha \neq 0$ for $x \neq 0$. Furthermore, $\delta(0) \neq 0$.