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How isotropic are turbulent flows generated by using periodic conditions in a cube?

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ABSTRACT

In numerical simulations, "isotropic" turbulent flows are always generated by using periodic conditions. We show that these periodic conditions mathematically lead to large-scale anisotropy which can be about 10% of the mean values, and thus prevent existing post-processing results from being accurate. A decomposition method by employing spherical harmonics is then proposed to distinguish this scale-dependent anisotropy effect from others and to minimize the related post-processing error.

1. Introduction

In order to numerically generate three-dimensional homogeneous isotropic turbulent (HIT) flows, the commonly-used method is to assume three-dimensional periodic conditions in a cube. There have been enormous literatures following this methodology, for instance Refs. [1–8]. These generated data are widely used in various topics, including the analysis of small-scale isotropic structures [9–12]. It is then quite important to know that in which range the turbulent flow can be isotropic. In particular, the upper bound of this isotropic range corresponds to the largest isotropic scale, which is usually considered to be the order of energy-containing scale or integral scale. However, to our knowledge, there is still no systematic discussion on this largest isotropic scale for the numerically generated turbulent flows with a cubic box and the periodic boundary condition applied in all directions.

Practical numerical calculations usually consider a cube with length $R = 2\pi$ in each direction. From existing direct numerical simulations (DNS) the integral scale is usually chosen to be smaller than 1.4 to avoid the influence of cube (see *e.g.* Refs. [5,2,13,14]). One of the few discussions on this influence can be found from Gotoh et al. [14], who remarked that the small number of energy-containing Fourier modes can cause large-scale anisotropy, which is measured by comparing the longitudinal and transverse structure functions. However, as will be analyzed in the present paper, this is a tensor-level anisotropy effect that is presented as redistribution among tensor components, whereas the lower-level anisotropy effect, which stems from periodic conditions, has not been systematically discussed.

In the present paper, we aim at discussing the large-scale influence of the periodic condition in a cube. By theoretical and numerical investigations, it will be shown that there always exists obvious anisotropy among the axis directions, the face diagonal directions and the cube diagonal directions, which even leads to different summation structure functions along these directions. We will show that this anisotropy effect differs from traditional tensor-level studies and is a congenital defect of numerical simulations. Decomposition by spherical harmonics functions will quantitatively show this large-scale anisotropy influence.

2. Descriptions of problem

In this section, we start from a 2D description and show that there is always anisotropy influence from periodic conditions to smaller scales. Unlike the discussions of Gotoh et al. [14], this anisotropy is expressed as a difference between the longitudinal structure functions in the axis directions and the face diagonal directions. It would also be easy to extend the following analysis to a 3D cube and to add cube diagonal directions into comparison.

See Fig. 1 as a sketch. The square $ABCD$ is a periodic domain in both directions. In the following we consider a scalar field and a vector field respectively to express the problem and to clarify the difference between the present problem and traditional studies.

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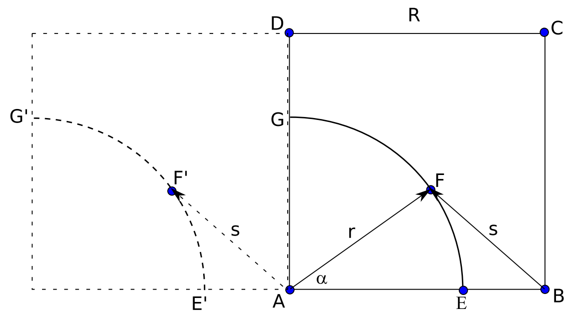

Fig. 1. Two-dimensional sketch for the anisotropy of longitudinal structure functions in the axis directions and the face diagonal directions.

2.1. Second-order structure function in a scalar field

We consider a scalar field Θ in the periodic domain and expect it to be homogeneous isotropic. From periodic conditions we have

$$\Theta(A) = \Theta(B) = \Theta(C) = \Theta(D). \tag{1}$$

Then we define the ensemble average operation as $\langle\rangle$. In particular, the second-order structure function at a two-point displacement is defined as

$$S(\boldsymbol{x}, \boldsymbol{r}) = \langle(\Theta(\boldsymbol{x} + \boldsymbol{r}) - \Theta(\boldsymbol{x}))^2\rangle. \tag{2}$$

Because of homogeneity,[1] it can also be rewritten as

$$S(\boldsymbol{r}) = \langle(\Theta(A + \boldsymbol{r}) - \Theta(A))^2\rangle. \tag{3}$$

Under ideal isotropy, we can calculate the second-order structure function of a two-point distance r by an average operation on a 1/4 circumference (which is also frequently employed in practical post-processing procedures):

$$S(r) = \frac{2}{\pi r}\int_0^{\pi/2} S(\boldsymbol{r}) r \mathrm{d}\alpha. \tag{4}$$

With the isotropy assumption, $S(\boldsymbol{r})$ does not depend on the angle α and can exit the integer, and this equation will be simplified to the identical equation $S(r) = S(r)$.

When the periodic conditions are involved, additional restrictions are implied at large scales. For example with the periodicity $\Theta(A) = \Theta(B)$, we can rewrite Eq. (3) as

$$S(\boldsymbol{r}) = \langle(\Theta(A + \boldsymbol{r}) - \Theta(B))^2\rangle. \tag{5}$$

Since $A + \boldsymbol{r} = B + \boldsymbol{s}$ we have

$$S(\boldsymbol{r}) = S(\boldsymbol{s}). \tag{6}$$

When we perform the average operation (4), this can also be regarded as an integration over the arc $E'G'$:

$$S(r) = \frac{2}{\pi}\int_0^{\pi/2} S(\boldsymbol{r} + (A - B)) \mathrm{d}\alpha. \tag{7}$$

Note that this equation is derived by using the periodic condition in the direction of AB. The periodic conditions in other directions can lead to other equations, but the discussions are similar. The assumption of isotropy implies that the statistical variables do not depend on the direction. Hence Eq. (7) can be simplified as

$$S(r) = \frac{2}{\pi}\int_0^{\pi/2} S(s) \mathrm{d}\alpha. \tag{8}$$

From the law of cosines $s = (R^2 + r^2 - 2Rr\cos\alpha)^{1/2}$, it leads to

$$S(r) = \frac{2}{\pi}\int_0^{\pi/2} S((R^2 + r^2 - 2Rr\cos\alpha)^{1/2}) \mathrm{d}\alpha. \tag{9}$$

With the initial condition $S(0) = 0$, one puts $r = 0$ in Eq. (9) and can immediately obtain $S(R) = 0$. Then, putting $r = R$ in Eq. (9) and by noting that Eq. (9) can be rewritten in the following formula when $r = R$:

$$0 = S(R) = \int_0^{\sqrt{2}R} S(s) w(s) \mathrm{d}s, \tag{10}$$

with a weight function $w(s) = 2s/(4R^2s^2 - s^4)^{1/2}$, which satisfies that $\forall 0 < s < 2R$, $w(s) > 0$. From Eq. (3) we also have $S(s) \geq 0$. Continuity of $S(s)$ then implies that $\forall 0 \leq s \leq \sqrt{2}R$, $S(s) \equiv 0$ (otherwise, supposing $S(t) > 0$ with $0 < t \leq \sqrt{2}R$, $\exists \epsilon > 0$, s.t.$\forall \tau \in (t - \epsilon, t + \epsilon)$, $S(\tau) > 0$, then $\int_0^{\sqrt{2}R} S(s)w(s)\mathrm{d}s \geq \int_{t-\epsilon}^{t+\epsilon} S(s)w(s)\mathrm{d}s > 0$ which conflicts with Eq. (10)) which is a non-physical trivial solution. In order to show the influence on a real field, we assume a classical scaling behavior $S(r) = r^{2/3}$, then the right hand side of Eq. (9) can be simplified as

$$S_{sl}(r) = \frac{2}{\pi r}\int_0^{\pi/2} (R^2 + r^2 - 2Rr\cos\alpha)^{1/3} r \mathrm{d}\alpha, \tag{11}$$

where S_{sl} means using the scaling law. Clearly, a paradox arises that $S_{sl}(r)$ is not of the order $r^{2/3}$. Fig. 2 shows the scaling behaviors of $S_{sl}(r)$, where $R = 1$. From Fig. 2(a) we find that $r^{-2/3}S_{sl}(r)$ is not constant; from Fig. 2(b) the scaling exponent of $S_{sl}(r)$ is not 2/3 either. Note that this is just a specific example to show that any isotropic scaling cannot be guaranteed, while for any other scalings the conclusions are the same. This obviously illustrates the necessity of investigating the influence of large-scale periodic condition.

Note that from Ref. [15] we can also find another effect of the domain size: when the domain size is not large enough (*e.g.* the order of integral scale), some intermittency is observed. By contrast, from the present study, we show that the periodical domain can also lead to (large-scale) anisotropy.

2.2. Second-order structure function in a vector field

Consider the structure functions in a vector field. Similarly to the previous subsection, we can define the summation $D_{ii}(\boldsymbol{r}) = \langle(u_i(A + \boldsymbol{r}) - u_i(A))(u_i(A + \boldsymbol{r}) - u_i(A))\rangle$ which is a scalar function. It is easy to show that the conclusions in the previous subsection are also satisfied for $D_{ii}(r)$. In particular, the periodic condition prevents $D_{ii}(r)$ from being isotropic.

In practice we are usually interested in the behavior of the longitudinal structure function, defined as $D_{ll}(\boldsymbol{r}) = \langle(u_l(A + \boldsymbol{r}) - u_l(A))^2\rangle$ with l the direction of $\boldsymbol{r}$, rather than the summation $D_{ii}(\boldsymbol{r})$. By defining the transverse structure function $D_{nn}(\boldsymbol{r}) = \langle(u_n(A + \boldsymbol{r}) - u_n(A))^2\rangle$ with n the normal direction to $\boldsymbol{r}$, the tensor-level isotropic condition yields

$$D_{nn}(r) = D_{ll}(r) + \frac{r}{2}\frac{dD_{ll}(r)}{dr}, \tag{12}$$

which implies the relation

[1] We remark that non-trivial homogeneous scalar fields can exist with periodic boundary conditions. For construction, we suppose that a scalar field Θ is periodic at time 0 and is advected by a non-zero constant velocity, then using time average instead of ensemble average can lead to the independence of $\boldsymbol{x}$ in Eq. (2).