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Hidden gauge symmetry in holomorphic models

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ABSTRACT

We study the effect of a hidden gauge symmetry on complex holomorphic systems. For this purpose, we show that intrinsically any holomorphic system has this gauge symmetry. We establish that this symmetry is related to the Cauchy–Riemann equations, in the sense that the associated constraint is a first class constraint only in the case that the potential be holomorphic. As a consequence of this gauge symmetry on the complex space, we can fix the gauge condition in several ways and project from the complex phase-space to real phase space. Different projections are gauge related on the complex phase-space but are not directly related on the real physical phase-space.

1. Introduction

In several instances, in physics it is natural to select complex variables to develop a theory. For example, in Conformal Field Theory in two dimensions the conformal transformations of the metric are equivalent to the Cauchy–Riemann equations for holomorphic functions. In this paper, we consider a generalization of this concept, in the sense that, we regard a system defined in the complex space and we show that this system possesses a gauge symmetry. This symmetry is trivial when is analyzed directly in the context of the complex variables $z = x + iy$, because it says directly that the transformation is null $\delta z = 0$, then all the holomorphic functions are invariant under these transformations. However, these transformations are not trivial if we consider that the real and imaginary parts of z are allowed to transform on the complex plane. We show that these transformations are gauge transformations generated by a first class constraint in the context of the Dirac's canonical method. Then by selecting a gauge condition we can map our complex system to different real systems. The interesting point is that these systems are related by a gauge transformation on the complex phase-space.

In Quantum Mechanics one of the fundamental postulates is that every measurable physical quantity $\mathcal{A}$ is described by an operator A acting on the state space; and this operator is an observable. A common hypothesis is to select Hermitian operators, in order to obtain measurable or observable quantities. This postulate implies that if we want to get all the information of the system, we must consider a complete set of commuting observables. Moreover, it has been hypothesized that some systems do not necessarily satisfy this postulate. Examples of these cases are: non-Hermitian models [1,2], with interesting applications as to generate entanglement in many-body systems [3]; the PT-symmetry [4,5], with striking applications in optics [6,7]. Also, we have theories with high order time derivatives as the Pais–Uhlenbeck model for particles [8–10] and Bernard–Duncan model for fields [11], noncommutative theories [12,13], higher order derivative theories of gravity [14,15] and complex theories of gravity [16].

There are several ways to address these models, for example when the Hermiticity is not available, it is natural to introduce a new kind of symmetry and in this way, the notion of PT-symmetry was introduced by Bender [4]. Furthermore, Ashtekar introduced a modification of the internal product, using the reality conditions, and this procedure also solves the problem in some cases [16]. Our approach is a generalization of the Ashtekar procedure, but written in a different way. Some years ago was shown that the reality conditions can be interpreted as second class constraints in the context of the Dirac's method of canonical quantization [17], and then the internal product is given in terms of the measure of the path integral with second class constraints. The object of this paper is to explore further this idea. We find that in any holomorphic theory there is intrinsically a gauge symmetry and the second class constraints of the Ashtekar formalism correspond to a selection of the gauge condition of the symmetry. However, there are many additional consistent gauge conditions. By selecting a gauge condition we get a different real physical system that is gauge related to another real system by a complex gauge transformation to be performed on the extended complex phase-space. In this form, we will show that using this gauge symmetry we can relate on the complex phase-space different real systems that are not related by

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a canonical transformation on the real phase-space. The work is organized as follows. In Section 2, we introduce the gauge symmetry and we show that is related to the Cauchy–Riemann equations. In Section 3 starting from the complex harmonic oscillator, and by using different gauges we obtain in the real physical space the potentials $k/2x^2$, ax^{-1}, bx^{-2} and $-ax^{-1}$. In Section 4 we generalize our construction to the two-dimensional complex space and we show that the harmonic oscillator and the Kepler problem are gauge related. In Section 5 we quantize the system using path integrals. Section 6 is devoted to our conclusions.

2. Complex theory for a first order theory

Let us consider a complex Lagrangian that is a function of the holomorphic coordinate $z = x + iy$ and their velocities

$$L(z,\dot{z}) = \frac{1}{2}\dot{z}^2 - V(z) \tag{1}$$

and we are assuming that the potential $V(z)$ is a holomorphic function of z, that is,

$$\frac{dV}{d\bar{z}} = 0 \tag{2}$$

Then, it is evident that the Lagrangian is invariant under the transformations

$$x' = x + \lambda(t), \qquad y' = y + i\lambda(t). \tag{3}$$

That leave z invariant, i.e. $\delta z = 0$. In consequence, from this point of view our system have a trivial symmetry. On the other hand, this symmetry is more useful if we decompose z in terms of real and imaginary parts. In this case, the Lagrangian is given by

$$L(z,\dot{z}) = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\dot{y}^2 + i\dot{x}\dot{y} - V_R(x,y) - iV_I(x,y). \tag{4}$$

The associated equations of motion of the above Lagrangian are not independent, since it is possible to divide them in real and imaginary parts and we get

$$\ddot{x} + \frac{\partial V_R(x,y)}{\partial x} = 0, \qquad \ddot{y} - \frac{\partial V_R(x,y)}{\partial y} = 0, \tag{5}$$

where it is clear that x and y are real quantities. Now, we proceed to develop the canonical formulation of this theory using the variables x and y. We select these variables, because in terms of holomorphic coordinates z, our symmetry is trivial in the sense we cannot establish, any relation between the holomorphic and anti-holomorphic coordinates. The canonical momenta for the Lagrangian (1), are

$$p_x := \frac{\partial L}{\partial \dot{x}} = \dot{x} + i\dot{y}, \qquad p_y := \frac{\partial L}{\partial \dot{y}} = -\dot{y} + i\dot{x}. \tag{6}$$

Using these definitions, we obtain the primary constraint

$$\Phi = 2p_{\bar{z}} = p_x + ip_y \approx 0, \tag{7}$$

where we introduce the weak equality symbol "≈" to emphasize that the quantity Φ is numerically restricted to be zero but does not identically vanish throughout phase space [18]. Following the usual definition of the canonical Hamiltonian in the phase-space

$$H = \dot{x}p_x + \dot{y}p_y - L, \tag{8}$$

we observe that H, L, p_x, p_y are complex quantities. Through the definition (8) we obtain the explicit total Hamiltonian

$$\begin{aligned} H_T &= \frac{p_z^2}{2} + V(z) + \mu\Phi \\ &= \frac{1}{2}p_x^2 + V_R(x,y) + iV_I(x,y) + \mu\Phi, \end{aligned} \tag{9}$$

where we add the primary constraint following the Dirac's method [19]. The resulting equations of motion are

$$\dot{x} = \{x, H_T\} = p_x + \mu, \qquad \dot{y} = \{y, H_T\} = p_y + i\mu, \tag{10}$$

$$\dot{p}_x = \{p_x, H_T\} = -\frac{\partial V}{\partial x}, \qquad \dot{p}_y = \{p_y, H_T\} = -\frac{\partial V}{\partial y}. \tag{11}$$

Using equations (10) we observe that there is a gauge freedom since we have an arbitrary Lagrange multiplier. It is important to note that the temporal evolution is not necessarily a real quantity, so our real variables x and y under evolution could obtain an imaginary contribution.

In the following, we are going to evolve the primary constraint (7) and in this way we should understand what kind of constraint is, first or second class. The primary constraint Φ evolves as

$$\dot{\Phi} = \{\Phi, H\} = -2\frac{dV(z)}{d\bar{z}} \approx 0. \tag{12}$$

In the present case, we observe that the temporal evolution of Φ imposes as a result the Cauchy–Riemann equations for $V(x,y)$ (see Eq. (2)). In consequence if $V(x,y)$ holomorphic function we obtain that Φ is first class constraint. Furthermore, there are no additional constraints and we will get as a result that our reduced phase-space has therefore two degrees of freedom. In the other hand, following the Dirac's quantization method the physical states are defined by imposing that the action of the first class constraint over the states is equal to zero. In the coordinate representation this we will imply

$$\left(-i\hbar\frac{\partial}{\partial x} + \hbar\frac{\partial}{\partial y}\right)[G_R(x,y) + iG_I(x,y)] = 0, \tag{13}$$

resulting the Cauchy–Riemann equations, if we decompose in real and imaginary parts. In this way, the Cauchy–Riemann equations appear in this formalism as an invariance under translations generated by the constraint. In other words, we obtain that our theory is compatible with the Dirac's formulation [19,18], but it must satisfy that the potential is a holomorphic function

$$\hat{\Phi}V(x,y) = 0. \tag{14}$$

Now, following the Dirac's conjecture this constraint will be the generator of gauge transformation [19]. The transformations produced by the first class constraint are

$$\delta x = \{x, \epsilon\Phi\} = \epsilon, \qquad \delta y = \{y, \epsilon\Phi\} = i\epsilon, \tag{15}$$

$$\delta p_x = \{p_x, \epsilon\Phi\} = 0, \qquad \delta p_y = \{p_y, \epsilon\Phi\} = 0. \tag{16}$$

In consequence, we get

$$\delta z = \{z, \Phi\} = 0, \tag{17}$$

in agreement with the transformations (3). From a pragmatic point of view z and p_z are Dirac's observables with null variation, but it implies a more complicated structure if we take in account the variation of the real and imaginary part of z. In this framework and if we pay attention to the equations (10), it is necessary to impose a gauge condition in order to obtain a real reduced phase-space. Then according to the gauge condition that we choose we can obtain a different real theory. The interesting point is that all these real theories are connected by a complex gauge transformation in the original extended phase-space. In fact, the equations of motion that we get from the Hamiltonian formulation are complex quantities obtained for the real and imaginary parts. Furthermore, the phase-space is wider than the configuration space since μ exists in this formulation, and it is possible to choose as a real quantity either δx or δy. For the purpose of applying a method that is not trivial using this mathematical structure, we must propose a gauge