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A self-adjoint arrival time operator inspired by measurement models

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ABSTRACT

We introduce an arrival time operator which is self-adjoint and, unlike previously proposed arrival time operators, has a close link to simple measurement models. Its spectrum leads to an arrival time distribution which is a variant of the Kijowski distribution (a re-ordering of the current) in the large momentum regime but is proportional to the kinetic energy density in the small momentum regime, in agreement with measurement models. A brief derivation of the latter distribution is given. We make some simple observations about the physical reasons for self-adjointness, or its absence, in both arrival time operators and the momentum operator on the half-line and we also compare our operator with the dwell time operator.

1. Introduction

The arrival time problem in quantum mechanics is the question of determining the probability that an incoming wave packet, for a free particle, arrives at the origin in a given time interval [1–5]. Classically, for a particle with initial position x and momentum p, the arrival time is given by the quantity

$$\tau = -\frac{mx}{p}. \tag{1}$$

The quantum problem is most simply solved using the spectrum of an operator corresponding to this quantity, such as that first studied by Aharonov and Bohm [6],

$$\hat{T}_{AB} = -\frac{m}{2}\left(\hat{x}\frac{1}{\hat{p}} + \frac{1}{\hat{p}}\hat{x}\right). \tag{2}$$

A heuristic result due to Pauli [7] (significantly updated by Galapon [8,9]) indicates that an object such as this, which is conjugate to a Hamiltonian with a semi-bounded spectrum, cannot be self-adjoint. Indeed we find that its eigenstates, which in the momentum representation (with $\hat{x} \to i\hbar\partial/\partial p$) are given by

$$\phi_\tau(p) = \left(\frac{|p|}{2\pi m\hbar}\right)^{\frac{1}{2}} e^{i\frac{p^2}{2m\hbar}\tau}, \tag{3}$$

are complete but not orthogonal. There is a POVM associated with these states [10] from which an arrival time distribution can be constructed and it coincides with that postulated by Kijowski [11], namely

$$\begin{aligned}\Pi_K(\tau) &= |\langle\psi|\phi_\tau\rangle|^2 \\ &= \frac{1}{m}\langle\psi_\tau||\hat{p}|^{\frac{1}{2}}\delta(\hat{x})|\hat{p}|^{\frac{1}{2}}|\psi_\tau\rangle\end{aligned} \tag{4}$$

(where $\Pi(\tau)d\tau$ is the probability of arriving at the origin between τ and $\tau+d\tau$), for which there is some experimental evidence [12]. This is related by a simple operator re-ordering to the quantum-mechanical current at the origin, $\langle\hat{J}(t)\rangle$, the result expected on classical grounds, where the current operator is given by

$$\hat{J}(t) = \frac{1}{2m}\left(\hat{p}\delta(\hat{x}(t)) + \delta(\hat{x}(t))\hat{p}\right), \tag{5}$$

with $\hat{x}(t) = \hat{x} + \hat{p}t/m$. This picture, the standard one, is nicely summarized in Ref. [13] and some developments of it and explorations of the underlying mathematics are described in Refs. [8,9,14,15].

The purpose of the present paper is to make two contributions to the standard picture presented above. The first is to discuss three simple self-adjoint modifications of the Aharonov–Bohm operator, discuss the relationship with the momentum operator on the half-line, and identify the underlying physically intuitive reasons why some of these operators are self-adjoint and some not. The second and main result is to present a new self-adjoint arrival time operator which has a much closer link to models of measurement than any previously studied operators.

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2. Self-adjoint arrival time operators

The lack of self-adjointness of Eq. (2) is not necessarily a problem but nevertheless a number of efforts have been made to restore it. Here, we take a simple approach and note that self-adjointness may be achieved by a number of simple modifications of the states Eq. (3). The states

$$\phi_\tau(p) = \left(\frac{|p|}{2\pi m\hbar}\right)^{\frac{1}{2}} e^{i\epsilon(p)\frac{p^2}{2m\hbar}\tau}, \tag{6}$$

where $\epsilon(p)$ is the sign function, are orthogonal and complete and so are eigenstates of a self-adjoint operator. This operator, first considered by Kijowski [11] and subsequently explored at length by Delgado and Muga [16], may be written

$$\hat{T}_{KDM} = -\frac{m}{2}\left(\hat{x}\frac{1}{|\hat{p}|} + \frac{1}{|\hat{p}|}\hat{x}\right), \tag{7}$$

and is a quantization of the classical expression $-mx/|p|$. It is also usefully written in the representation using the pseudo-energy, $\xi = p|p|/2m$, where we have

$$\hat{T}_{KDM} = -i\hbar\frac{\partial}{\partial\xi} \tag{8}$$

(which acts on states $\Phi(\xi) = (m/p)^{\frac{1}{2}}\phi(p)$), and is self-adjoint since ξ takes an infinite range. This is in contrast to the Aharonov–Bohm operator which, in the energy representation, has the form

$$\hat{T}_{AB} = \left(-i\hbar\frac{\partial}{\partial E}\right) \oplus \left(-i\hbar\frac{\partial}{\partial E}\right) \tag{9}$$

where the two parts of the direct sum refer to the positive and negative momentum sectors. Its lack of self-adjointness is due to the fact that $E > 0$, as is frequently noted. (See for example, Ref. [17].)

A second modification of the Aharonov–Bohm operator is to superpose opposite values of τ in Eq. (3) and then note that the subsequent states, which are proportional to $|p|^{\frac{1}{2}}\sin(p^2\tau/2m\hbar)$ are orthogonal and are the eigenstates of the self-adjoint operator

$$\hat{T}_{MI} = \sqrt{\hat{T}_{AB}^2}, \tag{10}$$

considered by de la Madrid and Isidro [18]. This is a quantization of $m|x|/|p|$. A third modification is to note that the orthogonality of the states $|p|^{\frac{1}{2}}\sin(p^2\tau/2m\hbar)$ is not affected by restriction to positive or negative momenta so we may consider these two sectors separately and as a consequence the operator

$$\hat{T}_3 = \theta(\hat{p})\hat{T}_{MI}\theta(\hat{p}) - \theta(-\hat{p})\hat{T}_{MI}\theta(-\hat{p}), \tag{11}$$

is self-adjoint. This operator, which does not seem to have been noted previously, is a quantization of the classical expression $m|x|/p$.

These three examples all side-step the Pauli theorem since they do not have canonical commutation with the Hamiltonian. Furthermore, they all give probability distributions which are simple variants on the Kijowski distribution, the expected result, as is easily deduced from their eigenstates.

From these three examples we make the following simple observation. The Aharonov–Bohm operator arises from the quantization of the classical expression $-mx/p$ and is not self-adjoint. However, quantizing any of the three classical expressions $-mx/|p|$, $m|x|/|p|$ or $m|x|/p$ leads to a self-adjoint operator. Hence, self-adjoint modifications of the Aharonov–Bohm operator are easily obtained by relinquishing just one or two bits of information, namely the sign of x, or p, or the signs of both. The relinquished information is essentially the specification of whether the particle is incoming (x and p with opposite signs) or outgoing (x and p with the same sign). Of course in practice we are usually interested in the arrival time for a given state, for which this information is already known, at least semiclassically, so from this point of view the difference between the Aharonov–Bohm operator and its self-adjoint variants may not be important. Nevertheless it is of interest to uncover the underlying origins of self-adjointness or its absence and the above properties give some useful clues.

Physically speaking, self-adjointness or its absence are about precision. A self-adjoint operator has orthonormal eigenstates and an associated projection operator onto a range of its spectrum. Projections onto different ranges have zero overlap. An operator that is not self-adjoint has non-orthogonal eigenstates and has at best a POVM onto a range of its spectrum. Two POVMs localizing onto different ranges will have a small overlap which means there is intrinsic imprecision in the specification the ranges they localize onto. To understand the origin of the lack of self-adjointness in the Aharonov–Bohm operator we would like to find a physically intuitive understanding of where this imprecision comes from.

3. The momentum operator on the half-line

To understand the above issue, we turn to the frequently-studied situation of the momentum operator on the half-line $x > 0$ [19–21]. There, the momentum operator cannot be made self-adjoint since it generates translations into negative x. However, $\hat{p}^2$ can be made self-adjoint, with suitable boundary conditions, and therefore, by the spectral theorem, $|\hat{p}|$ can be made self-adjoint. Hence just by relinquishing information about the sign of $\hat{p}$ a self-adjoint operator is obtained. The obstruction to self-adjointness on the half-line therefore lies in the sign function of $\hat{p}$. Differently put, the problem is that the operator $\theta(\hat{x})\theta(\hat{p})\theta(\hat{x})$ cannot be made self-adjoint. For similar reasons, we also note that the position operator on the positive momentum sector cannot be self-adjoint. A POVM for the momentum operator on the half-line may be constructed, but this is not directly relevant to what we do here [21].

We propose that there is a simple physical way of understanding the underlying imprecision linked to this lack of self-adjointness. Suppose we tried to measure the momentum. Let us therefore consider a simple measurement model of momentum on the half-line $x > 0$ using sequential position measurements, from which information about momentum can be deduced. Similar approaches to calculating the time-of-flight momentum have been given elsewhere [22] and we make use of these results, but adapted to the case of propagation in the region $x > 0$. We suppose we have an initial incoming state ψ at time t_0 consisting of a spatially very broad gaussian, close to a plane wave, of momentum $p_0 < 0$ and we ask if it passes through a spatial region $[\bar{x}_1 - \Delta, \bar{x}_1 + \Delta]$ in $x > 0$ at time t_1 and at a later time t_2 through a spatial region $[\bar{x}_2 - \Delta, \bar{x}_2 + \Delta]$. The probability for these two measured results is

$$p(\bar{x}_1, t_1, \bar{x}_2, t_2) = \langle\psi|g^\dagger(t_1, t_0)P_{\bar{x}_1}g^\dagger(t_2, t_1)P_{\bar{x}_2}g(t_2, t_1)P_{\bar{x}_1}g(t_1, t_0)|\psi\rangle \tag{12}$$

where

$$P_{\bar{x}} = \int_{\bar{x}-\Delta}^{\bar{x}+\Delta} dx|x\rangle\langle x| \tag{13}$$

is a projector onto the range $[\bar{x} - \Delta, \bar{x} + \Delta]$ and $g(t_1, t_0)$ denotes the propagator in the region $x > 0$. The precise form of the propagator depends on the boundary conditions on the states imposed at $x = 0$. There is a one-parameter family that leads to a self-adjoint Hamiltonian, of the form