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New 1-step extension of the Swanson oscillator and superintegrability of its two-dimensional generalization

Bijan Bagchi [a], Ian Marquette [b,*]

[a] Department of Applied Mathematics, University of Calcutta, 92 Acharya Prafulla Chandra Road, Kolkata 700 009, India
[b] School of Mathematics and Physics, The University of Queensland, Brisbane, QLD 4072, Australia

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ABSTRACT

We derive a one-step extension of the well known Swanson oscillator that describes a specific type of pseudo-Hermitian quadratic Hamiltonian connected to an extended harmonic oscillator model. Our analysis is based on the use of the techniques of supersymmetric quantum mechanics and addresses various representations of the ladder operators starting from a seed solution of the harmonic oscillator expressed in terms of a pseudo-Hermite polynomial. The role of the resulting chain of Hamiltonians related to similarity transformation is then exploited. In the second part we write down a two dimensional generalization of the Swanson Hamiltonian and establish superintegrability of such a system.

1. Introduction

Seeking rational extensions of known solvable systems is currently one of the major topics of research in quantum mechanics. Indeed, recent times have witnessed a great deal of activity in generating rationally extended models in the context of polynomial Heisenberg algebras (PHA) and exceptional orthogonal polynomials [1–4]. A particularly simple one is for the problem of harmonic oscillator in which case the ladder operators have been constructed through the combination of the oscillator creation and annihilation operators along with the supercharges or having the combination of the latter. In this regard, harmonic rational extension has been carried out in the framework of a supersymmetric quantum mechanics (SUSYQM) theory [5,6]. In particular, 1-step extensions have been sought for the radial oscillator and its generalization, the Scarf I (also sometimes referred to as Poschl–Teller or Poschl–Teller I) [7–15], and the generalized Poschl–Teller (also sometimes referred to as hyperbolic Poschl–Teller or Poschl–Teller II) [9,10,16].

In this article we apply standard supersymmetric (SUSY) techniques to develop a systematic procedure for obtaining a solvable rational extension of a non-Hermitian quadratic Hamiltonian that was proposed by Swanson [17] sometime ago and was shown to possess real and positive eigenvalues. Such a model that we present in its differential operator form is new and opens the way of constructing new classes of non-Hermitian quadratic Hamiltonians based on rational extensions. We also address a general two-dimensional analog of Swanson Hamiltonian from a two-dimensional perspective that is separable in Cartesian coordinates and establish superintegrability of such a system by making suitable use an underlying ladder operators.

2. Swanson model

The Swanson model deals with a specific type of a non-Hermitian Hamiltonian connected to an extended harmonic oscillator problem. A general quadratic form for it has the simple structure

$$H^s = \omega a^\dagger a + \alpha a^2 + \beta (a^\dagger)^2 + \frac{1}{2}\omega \tag{2.1}$$

where a and $a^\dagger$ are respectively the usual annihilation and creation operators of the one-dimensional harmonic oscillator obeying the canonical commutation relation $[a, a^\dagger] = 1$. In (2.1), ω, α and β are real constants. It is clear that with $\alpha \neq \beta$, H^s ceases to be Hermitian. Nonetheless, it is pseudo-Hermitian [18] embodying parity-time symmetry and supports a purely real, positive spectrum over a certain range of parameters. Swanson Hamiltonian has been widely employed as a toy model to investigate a wide class of non-Hermitian systems for different situations. Some of its applications

* Corresponding author.
E-mail addresses: bbagchi123@gmail.com (B. Bagchi), i.marquette@uq.edu.au (I. Marquette).

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have been in exploring the choice of a unique and physical metric operator to generate an equivalent Hermitian system [18–22], setting up of a group structure of the Hamiltonian [23,24], looking for possible quasi-Hermitian and pseudo-supersymmetric (SUSY) extensions [25–27], working out an $\mathcal{N}$-fold SUSY connection [28], estimating minimum length uncertainty relations that result from the structure of non-commutative algebras [29,30], studying a relevant R-deformed algebra [31], writing down supercoherent states [32] and investigating some of the aspects of classical and quantum dynamics for it [33,34].

A similarity transformation allows us to write down [19] the Hermitian equivalence of H^s. In this way an equivalent Hermitian counterpart of H^s turns out to be a scaled harmonic oscillator:

$$h = \rho H \rho^{-1}$$
$$= -\frac{1}{2}(\omega - \alpha - \beta)\frac{d^2}{dx^2} + \frac{1}{2}x^2\frac{\omega^2 - 4\alpha\beta}{\omega - \alpha - \beta} + \frac{1}{2}\omega \tag{2.2}$$

where $\rho = e^{\frac{1}{2}\lambda x^2}$ along with the accompanying eigenfunctions

$$\psi_n(x) = N_n e^{-\frac{1}{2}x^2(\lambda+\Delta^2)} H_n(\Delta x). \tag{2.3}$$

In (2.2) and (2.3) the parameters λ and Δ are defined by

$$\lambda = \frac{\beta - \alpha}{\omega - \alpha - \beta}, \quad \Delta = \frac{(\omega^2 - 4\alpha\beta)^{\frac{1}{4}}}{(\omega - \alpha - \beta)^{\frac{1}{2}}} \tag{2.4}$$

and H_n is an nth degree Hermite polynomial. The eigenfunctions are orthonormal with respect to the quantity $e^{\lambda x^2}$ i.e.

$$\int \psi_m^*(x) e^{\lambda x^2} \psi_n(x) dx = \delta_{mn}. \tag{2.5}$$

We note that the scaled harmonic oscillator Hamiltonian h as given by (2.2) can be cast into the standard form through the transformation

$$h \to \tilde{h} = \frac{2}{\sqrt{\omega^2 - 4\alpha\beta}}(h - \frac{1}{2}\omega) \tag{2.6}$$

and introducing a change of variable $x \to z$ as given by

$$x \to z = \sqrt[4]{\frac{\omega^2 - 4\alpha\beta}{(\omega - \alpha - \beta)^2}}\, x \equiv \Delta x. \tag{2.7}$$

Thus we arrive at the following Schrödinger operator for $\tilde{h}$:

$$\tilde{h} = -\frac{d^2}{dz^2} + z^2 \tag{2.8}$$

from which we develop a SUSY scheme by means of standard supercharges that go with it.

3. SUSY scenario

The Hamiltonian $\tilde{h}$ can be embedded in a supersymmetric setting [6] by defining a pair of partner Hamiltonians in terms of the z-coordinate

$$\tilde{h}^{(+)} = \tilde{A}^\dagger \tilde{A} = -\frac{d^2}{dz^2} + \tilde{V}^{(+)} - \tilde{E} \equiv \tilde{h} - \tilde{E} \tag{3.9}$$

$$\tilde{h}^{(-)} = \tilde{A}\tilde{A}^\dagger = -\frac{d^2}{dz^2} + \tilde{V}^{(-)} - \tilde{E} \tag{3.10}$$

where the operators $\tilde{A}$ and $\tilde{A}^\dagger$ are governed by the superpotential $\tilde{W}(z)$:

$$\tilde{A}^\dagger = -\frac{d}{dz} + \tilde{W}(z), \quad \tilde{A} = \frac{d}{dz} + \tilde{W}(z). \tag{3.11}$$

This provides identification of the corresponding partner potentials

$$\tilde{V}^{(\pm)}(z) = \tilde{W}^2(z) \mp \tilde{W}'(z) + \tilde{E}. \tag{3.12}$$

It should be remarked that the component Hamiltonians $\tilde{h}^{(+)}$ and $\tilde{h}^{(-)}$ intertwine with the operators $\tilde{A}$ and $\tilde{A}^\dagger$ in the manner $\tilde{A}\tilde{h}^{(+)} = \tilde{h}^{(-)}\tilde{A}$ and $\tilde{A}^\dagger\tilde{h}^{(-)} = \tilde{h}^{(+)}\tilde{A}^\dagger$. Further the underlying nodeless eigenfunction $\tilde{\phi}(x)$ of the Schrödinger equation

$$(-\frac{d^2}{dz^2} + \tilde{V}^{(+)})\tilde{\phi}(z) = \tilde{E}\tilde{\phi}(z) \tag{3.13}$$

has the feature that it is given by $\tilde{W}(z) = -(\log(\tilde{\phi}(z))'$ where the prime stands for the derivative with respect to z. The factorization energy $\tilde{E}$ is assumed to be smaller than or equal to the ground-state energy of $\tilde{V}^{(+)}$. From (2.8) and (3.9), it is clear that $\tilde{V}^{(+)}$ is identifiable with $\tilde{V}^{(+)} = z^2$. Then with $\tilde{E} = 1$ and $\tilde{W}(z) = z$, the partner potential turns out to be $\tilde{V}^{(-)} = z^2 + 2$ reflecting the shape-invariance character of the harmonic oscillator, $\tilde{V}^{(-)}$ being just a translated oscillator with respect to $\tilde{V}^{(+)}$.

If however, $\tilde{E} < 1$, then the only possible nodeless seed solutions of (3.13) are of the type

$$\tilde{\phi}_m(z) = \mathcal{H}_m(z) e^{\frac{1}{2}z^2}, \quad m = 2, 4, 6, \ldots \tag{3.14}$$

where the pseudo-Hermite polynomial $\mathcal{H}_m(z)$ is related to the standard Hermite by $\mathcal{H}_m(x) = (-1)^m H_m(ix)$. We remark that ψ_m of the equivalent Hamiltonian representation of Swanson Hamiltonian is invariant under $x \to ix$. This means that given the correspondence between the Hermite polynomials and their pseudo-Hermite counterparts, it follows from (2.3) that

$$\psi_n(ix) = N_n e^{\frac{1}{2}x^2(\lambda+\Delta^2)} \mathcal{H}_n(i\Delta x), \quad n = 0, 1, 2, 3, \ldots \tag{3.15}$$

is also an eigenstate of H^s. Now transforming $\psi_n(ix)$ as $\psi_n(ix) \to \rho^{-1}\psi_n(ix)$ we in fact recover the seed solution $\psi_n(ix) = N_n e^{\frac{1}{2}x^2(\lambda+\Delta^2)} H_n(ix)$ where $H_n(ix)$ is the standard Hermite polynomial.

Given $\tilde{\phi}_m(z)$ as (3.14), the accompanying superpotential $\tilde{W}(z)$ is given by

$$\tilde{W}(z) = -z - \frac{\mathcal{H}_m'}{\mathcal{H}_m} \quad (\equiv -\frac{\tilde{\phi}'}{\tilde{\phi}}). \tag{3.16}$$

The partner potential $V\tilde{(-)}(z)$ then reads

$$\tilde{V}^{(-)}(z) = z^2 - 2[\frac{\mathcal{H}_m''}{\mathcal{H}_m} - (\frac{\mathcal{H}_m'}{\mathcal{H}_m})^2 + 1] \tag{3.17}$$

along with the energy spectrum

$$\tilde{E}_m = -2m - 1. \tag{3.18}$$

The explicit forms first appear in [35] in the course of deriving the exact closed form solutions of a generalized one-dimensional potential [36] that has a form intermediate to the harmonic and isotonic oscillators. Indeed the latter corresponds to $m = 2$ which is the second category of rational extension. Subsequently, a translational shape invariant property of (3.16) with a zero translational amplitude was established in [37] and also more recently discussed in the Krein–Adler and Darboux–Crum construction of these systems [38,39].

At this stage it is instructive to revert to the x-coordinate and write down the supersymmetric partner Hamiltonian counterpart of H^s. Defining

$$J = \frac{2}{\sqrt{\omega^2 - 4\alpha\beta}}, z = \Delta x, \quad E = \frac{\tilde{E}}{J} \tag{3.19}$$

and noting that $\tilde{h}^{(+)}$ represents the form of (2.2), the $\tilde{h}^{(+)}$ component reads in terms of the parameters J and Δ: