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An update on the classical and quantum harmonic oscillators on the sphere and the hyperbolic plane in polar coordinates

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ABSTRACT

A simple derivation of the classical solutions of a nonlinear model describing a harmonic oscillator on the sphere and the hyperbolic plane is presented in polar coordinates. These solutions are then related to those in cartesian coordinates, whose form was previously guessed. In addition, the nature of the classical orthogonal polynomials entering the bound-state radial wavefunctions of the corresponding quantum model is identified.

1. Introduction

During many years, there has been a continuing interest for some generalizations [1–4] of a classical nonlinear oscillator [5,6], which was introduced as a one-dimensional analogue of some quantum field theoretical models, and for the corresponding extensions [2,3,7–11] of its quantum version [12,13]. Such a model is indeed an interesting example of a system with nonlinear oscillations with a frequency showing amplitude dependence. Furthermore, since it contains both a nonlinear potential and a position-dependent mass, it is amenable to applications in those areas wherein the harmonicity of vibrations breaks down, such as in high-energy molecular states, or wherein a position-dependent effective mass is a useful concept, such as in many condensed-matter systems or many-body problems.

In 2004, Cariñena, Rañada, Santander, and Senthilvelan introduced a two-dimensional (and more generally n-dimensional) classical generalization [1] of the one-dimensional model of [5,6]. They established that the nonlinearity parameter λ, entering the definition of the potential and the position-dependent mass, can be interpreted as $-\kappa$, where κ is the curvature of the two-dimensional space, so that their model actually describes a harmonic oscillator on the sphere (for $\lambda = -\kappa < 0$) and on the hyperbolic plane (for $\lambda = -\kappa > 0$). They presented the solutions of the Euler–Lagrange equations in cartesian coordinates, showed that the system is superintegrable and that the Hamilton–Jacobi equations are separable in three different coordinate systems. Later on, the quantum version of this classical model was also exactly solved in the corresponding three coordinate systems, wherein the Schrödinger equation is separable [7,8].

In the present Letter, we deepen our understanding on these two-dimensional classical and quantum models by providing an update on their analysis in polar coordinates.

In [1], the Euler–Lagrange equations in cartesian coordinates were so complicated that they could not be directly solved in a simple way. Some particular expressions were then assumed for the solutions and the undetermined parameters they contained were shown to satisfy some constraints by inserting such expressions in the equations. Here we plan to prove that, in contrast, the Euler–Lagrange equations in polar coordinates simplify considerably, so that their solutions can be systematically derived.

Furthermore, in the solutions of the quantum model in polar coordinates [8], the precise nature of the classical orthogonal polynomials entering the bound-state radial wavefunctions was not determined. We will complete this analysis here, thereby extending to two dimensions a recent study [10], wherein the quantum one-dimensional model of [12] and [13] was re-examined.

2. Solutions of the Euler–Lagrange equations in polar coordinates

In cartesian coordinates x, y, the Lagrangian of [1] can be written as

$$L = \frac{1}{2}\frac{1}{1+\lambda(x^2+y^2)}[\dot{x}^2+\dot{y}^2+\lambda(x\dot{y}-y\dot{x})^2] - \frac{1}{2}\frac{\alpha^2(x^2+y^2)}{1+\lambda(x^2+y^2)}, \tag{1}$$

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where λ may be positive or negative and α is some real constant that we may assume positive. In polar coordinates, it can be rewritten as

$$L = \frac{1}{2}\left(\frac{\dot r^2}{1+\lambda r^2} + \frac{J^2}{r^2}\right) - \frac{1}{2}\frac{\alpha^2 r^2}{1+\lambda r^2}, \tag{2}$$

where the angular momentum $J = x\dot y - y\dot x = r^2\dot\varphi$ is a constant of the motion.

Considering now the Euler–Lagrange equations, we get a single differential equation to solve, namely

$$\ddot r - \frac{\lambda r}{1+\lambda r^2}\dot r^2 + \frac{\alpha^2 r}{1+\lambda r^2} - J^2\frac{1+\lambda r^2}{r^3} = 0, \tag{3}$$

since the constancy of J ensures that the other equation with respect to φ is automatically satisfied. To solve (3), we proceed in two steps.

First, on setting $\dot r = p(r)$, we obtain a first-order equation for p^2,

$$\frac{dp^2}{dr} - \frac{2\lambda r}{1+\lambda r^2}p^2 + \frac{2\alpha^2 r}{1+\lambda r^2} - 2J^2\frac{1+\lambda r^2}{r^3} = 0, \tag{4}$$

whose general solution is given by

$$p^2(r) = C(1+\lambda r^2) - \frac{J^2}{r^2} + \frac{\alpha^2}{\lambda} - \lambda J^2, \tag{5}$$

in terms of some integration constant C. Second, from (5), we get the differential equation

$$2dt = \frac{dr^2}{\sqrt{a + br^2 + cr^4}},$$
$$a = -J^2, \qquad b = C + \frac{\alpha^2}{\lambda} - \lambda J^2, \qquad c = C\lambda, \tag{6}$$

which can be easily integrated by taking into account the sign of the discriminant $\Delta = 4ac - b^2$ whenever $c \neq 0$ [14]. The solutions for $t = t(r^2)$ can then be inverted to yield $r^2 = r^2(t)$.

Finally, the integration of the first-order differential equation $\dot\varphi = J/r^2(t)$ [14] provides the functions $\varphi = \varphi(t)$ for $J \neq 0$ in terms of some constant K (since for $J = 0$, φ remains constant).

To write some physically-relevant results, it is worth observing that the value of the integration constant C is directly related to the energy E of the system. The latter is indeed given by

$$E = \frac{1}{2}\frac{1}{1+\lambda r^2}\left[\dot r^2 + \alpha^2 r^2 + \frac{J^2}{r^2}(1+\lambda r^2)\right] \tag{7}$$

and insertion of (5) in (7) leads to

$$E = \frac{1}{2}C + \frac{\alpha^2}{2\lambda} \qquad \text{or} \qquad C = 2E - \frac{\alpha^2}{\lambda}. \tag{8}$$

On the other hand, Eq. (7) can be rewritten as

$$E = \frac{1}{2}\frac{\dot r^2}{1+\lambda r^2} + V_{\rm eff}(r), \qquad V_{\rm eff}(r) = \frac{1}{2}\frac{\alpha^2 r^2}{1+\lambda r^2} + \frac{J^2}{2r^2}, \tag{9}$$

where the constancy of J allows us to group the term $J^2/(2r^2)$, coming from the kinetic energy, with the potential $V(r) = \alpha^2 r^2/[2(1+\lambda r^2)]$ to define an effective potential $V_{\rm eff}(r)$.

The possible values of E, and consequently of C, are determined by the behaviour of $V_{\rm eff}(r)$, where for $\lambda > 0$, r varies on the interval $(0, +\infty)$, while for $\lambda < 0$, it is restricted to $\left(0, 1/\sqrt{|\lambda|}\right)$. According to whether $J = 0$ or $J \neq 0$, $V_{\rm eff}(r)$ goes to 0 or $+\infty$ for $r \to 0$. On the other hand, $V_{\rm eff}(r)$ goes to $\alpha^2/(2\lambda)$ for $r \to \infty$ if $\lambda > 0$ or to $+\infty$ for $r \to 1/\sqrt{|\lambda|}$ if $\lambda < 0$. Moreover, it can be easily shown that for $J \neq 0$, $V_{\rm eff}(r)$ has a minimum $V_{\rm eff,min} = \frac{1}{2}|J|(2\alpha - \lambda|J|)$ at $r_{\rm min} = \sqrt{|J|/(\alpha - \lambda|J|)} \in (0, +\infty)$ or $\left(0, 1/\sqrt{|\lambda|}\right)$ (according to which case applies). Note that in the $\lambda > 0$ case, such a minimum only exists for J values such that $|J| < \alpha/\lambda$, thereby showing that bounded trajectories are restricted to low angular momentum values. It is worth pointing out that such a limitation on bounded motions for $\lambda > 0$ was not reported in [1] and that for $J = 0$, one may set $V_{\rm eff,min} = 0$. In Figs. 1 and 2, some examples are plotted for $\lambda > 0$ and $\lambda < 0$, respectively.

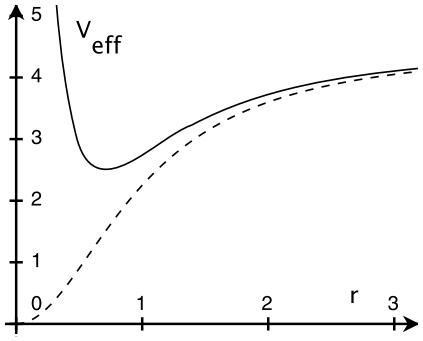

Fig. 1. Plot of $V_{\rm eff}(r)$, $\alpha = 3$, $\lambda = 1$, as a function of r for $J = 0$ (dashed line) and $J = 1$ (solid line).

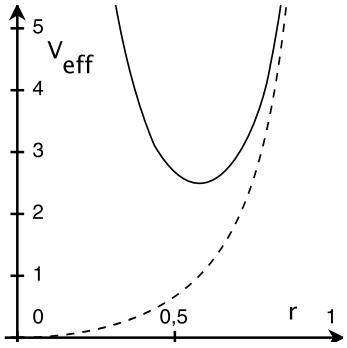

Fig. 2. Plot of $V_{\rm eff}(r)$, $\alpha = 2$, $\lambda = -1$, as a function of r for $J = 0$ (dashed line) and $J = 1$ (solid line).

The restrictions on the constants C, c, and $\Delta = -(2E + \lambda J^2 - 2\alpha J)(2E + \lambda J^2 + 2\alpha J)$ of Eq. (6) for each energy domain are given by

$$\begin{aligned}
&\text{(i)} && \text{If } \lambda > 0 \text{ and } V_{\rm eff,min} < E < \alpha^2/(2\lambda)\\
&&& \text{or if } \lambda < 0 \text{ and } V_{\rm eff,min} < E < +\infty,\\
&&& \text{then } -(\alpha - \lambda|J|)^2/\lambda < C < 0,\ c < 0, \text{ and } \Delta < 0;\\
&\text{(ii)} && \text{If } \lambda > 0 \text{ and } E = \alpha^2/(2\lambda), \text{ then } C = 0 \text{ and } c = 0;\\
&\text{(iii)} && \text{If } \lambda > 0 \text{ and } \alpha^2/(2\lambda) < E < +\infty,\\
&&& \text{then } 0 < C < +\infty,\ c > 0, \text{ and } \Delta < 0.
\end{aligned} \tag{10}$$

For $\lambda > 0$ and $V_{\rm eff,min} < E < \alpha^2/(2\lambda)$ or $\lambda < 0$, the complete solution is given by

$$r^2 = A\sin(2\omega t + \phi) + B, \quad B - A \le r^2 \le B + A,$$
$$A = \frac{1}{2|\lambda|\omega^2}\sqrt{[(\alpha - \lambda J)^2 - \omega^2][(\alpha + \lambda J)^2 - \omega^2]},$$
$$B = \frac{\alpha^2 - \lambda^2 J^2 - \omega^2}{2\lambda\omega^2}, \quad \phi \in [0, 2\pi),$$
$$\tan(\varphi - K) = \frac{\omega}{J}\left[B\tan\left(\omega t + \frac{\phi}{2}\right) + A\right] \quad \text{if } J \neq 0,$$
$$\varphi = K \quad \text{if } J = 0, \tag{11}$$