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Hall viscosity: A link between quantum Hall systems, plasmas and liquid crystals

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ABSTRACT

In this Letter, the assumption of two simple postulates is shown to give rise to a Hall viscosity term via an action principle formulation. The rationale behind the two postulates is clearly delineated, and the connections to an intrinsic angular momentum are emphasized. By employing this methodology, it is shown that Hall viscosity appears in a wide range of fields, and the interconnectedness of quantum Hall systems, plasmas and nematic liquid crystals is hypothesized. Potential avenues for experimental and theoretical work arising from this cross-fertilization are also indicated.

1. Introduction

The Navier–Stokes equations have been in existence for nearly 200 years, and are familiar to nearly all physicists. One of the central features of the Navier–Stokes equations is the existence of viscosity and the corresponding viscous stress tensor. A particularly elegant derivation of the same can be found in [1]. By building in rotational and time invariance, it can be shown that the viscous stress tensor has the form

$$N_{ijkl} = \eta\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) + \left(\zeta - \frac{2}{d}\eta\right)\delta_{ij}\delta_{kl}, \tag{1}$$

where d is the number of spatial dimensions of the system [2]. The tensor is symmetric under $(i\,j) \leftrightarrow (k\,l)$. If one breaks the time invariance symmetry, it is also possible to have a component that is *odd*, i.e. it has the property $N^A_{ijkl} = -N^A_{klij}$. We denote this component by N^A and the symmetric component by N^S. In general, $N^A = 0$ if rotational invariance is still to be preserved, but the case with $d = 2$ is special. By breaking parity (and time) invariance, it can be shown that a non-zero tensor N^A of the form

$$N^A_{ijkl} = \eta_H\left(\epsilon_{il}\delta_{kj} - \epsilon_{kj}\delta_{il}\right) \tag{2}$$

can be constructed [3] that retains rotational invariance. The coefficient η_H has been labeled the Hall viscosity or the odd viscosity [2] in scientific literature, and we shall adopt the former convention.

In this Letter, we shall show that the Hall viscosity can arise in a diverse array of unconnected fields such as quantum Hall systems, plasmas and nematic liquid crystals. We shall demonstrate that these fields are united through the existence of intrinsic angular momentum (driven by various mechanisms), allowing for the existence of a common mathematical structure. We commence with a brief summary of these three fields, then present the mathematical derivation, and identify the crucial physics questions that arise along the way. These questions are resolved subsequently, and we shall point out the similarities (and the differences) between these seemingly disparate fields.

The importance of Hall viscosity in the context of condensed matter was first noticed in [4], where it was proved that the coefficient η_H was proportional to the magnetic field in quantum Hall systems. In a wide variety of systems, it has been shown that the ratio of the Hall viscosity to the density gives rise to a constant, which has topological properties, and an intuitive physical interpretation – it represents the intrinsic angular momentum per particle [5,6]. The tensor N^A also possesses several other unusual properties, the chief of which is that it serves as a dissipationless tensor, i.e. it conserves energy. Hall viscosity has been studied extensively in current times, in a wide range of contexts, and we refer the reader to [7,8] for discussions of the same. In particular, it has been explored in the context of gauge/gravity duality [9], quantum Hall states [5,10], topological insulators [11], the Wess–Zumino term [12], and several others. In principle, the effects arising from Hall viscosity should be detectable via inhomogeneous electric fields or photon X-ray diffraction [7]. However, it is noteworthy that no such experimental realizations do actually exist thus far.

Let us now turn our attention to a field of physics that has been, and still is, studied extensively on an experimental basis – plasma physics. Since plasmas consist of charged particles, which exhibit the well-known phenomenon of Larmor gyration, effects

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arising from this phenomenon need to be taken into account when constructing physical models. These models are collectively described as Finite Larmor Radius (FLR) models, and are usually studied in the context of kinetic theory. These effects also percolate down to the level of fluid theories, described in the seminal works [13] and [14]. Of these, Braginskii's work [14] has been used extensively in modeling FLR effects, and numerical experiments establishing its successes and limitations are well documented.

A third area, unconnected to plasmas, that we shall explore in this Letter is the field of nematodynamics, which is commonly used in modeling liquid crystals. This field arose from the central works of Ericksen [15] and Leslie [16], collectively dubbed the Leslie–Ericksen (LE) equations. The LE equations are akin to the usual hydrodynamic equations, but they contain an additional dynamical variable, the director $\mathbf{n}$, that describes the direction of molecular alignment. The director introduces additional degrees of freedom, leading to the presence of a non-zero rotational kinetic energy, amongst other effects. To model the additional degrees of freedom, the need for spin (in the classical sense) as an independent dynamical variable was stated in [17], and a set of complete equations were presented in [18], which reduced to the LE equations in a suitable limit. The connections between hydrodynamical variants of liquid crystals, and their connections with conventional hydrodynamics have been explored in several works, see e.g. [19, 20].

The outline of the paper is as follows. In Section 2, we shall construct a simple action, and derive the relevant equations of motion. We also demonstrate therein how the aforementioned areas of physics can be treated on a unified footing through simple physical principles. We present the associated Hamiltonian formulation in Section 3, and discuss potential extensions of our approach. Finally, we summarize our results in Section 4. The genesis of our work, particularly in the context of intrinsic angular momentum, stems from the work undertaken in [21].

2. A unified action principle formulation

We commence with a construction of the action, and a derivation of the underlying equations of motion. Next, we address two important postulates used in constructing the action, and show how they arise as a result of concrete physical reasoning.

2.1. The action principle and the equations of motion

We shall choose to work with an action principle formalism, and employ the notation used in fluids and plasmas outlined in [22]. We shall work in two (spatial) dimensions and assume Cartesian geometry. Our starting point must be the definition of the relevant dynamical variables. It is clear that the bulk velocity of the fluid $\mathbf{v}$, and the density ρ constitute two such variables. In addition to ρ, we also specify a second thermodynamic variable – the entropy s. As our fourth variable, we choose ℓ, endowed with the dimensions of angular momentum density. It must be noted that ℓ may serve as an independent dynamical variable, or be a thermodynamic function of ρ and s. It is natural to wonder as to how and why ℓ arises, and we dub this *Issue I*, and return to it eventually.

There are two common methods for constructing action principles. The first is to work in a *Lagrangian* setting, wherein the fluid is modeled as a collection of 'particles' and the only time-varying field is the particle trajectory $\mathbf{q}(\mathbf{a}, t)$ with '$\mathbf{a}$' serving as the continuum label. The other fields, such as density, entropy, etc., are tied into the fluid 'particle' via suitable conservation laws. A second strategy entails the use of the *Eulerian* picture, which is the one that we shall use, owing to its higher usage in most fields of physics. In the Eulerian picture, the position $\mathbf{r}$ is fixed, and all the fields such as density, entropy, etc. are functions of $\mathbf{r}$ and t. When working within the Eulerian picture, an additional tool is required – constrained (or induced) variations. The reason stems from the fact that some of the Eulerian fields obey inherent conservation laws. Hence, when one resorts to the principle of least action and performs the variation, some of these fields must be varied in a constrained manner, i.e. the variations must preserve the underlying conservation laws. A detailed description of this method is presented in [23], and was recast into a more mathematical form by [24], who also dubbed it the Euler–Poincaré method.

Next, we need to decide upon the relevant conservation laws for these variables. Note that the constraint of mass conservation leads to the density obeying the continuity equation, i.e. one has

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \tag{3}$$

Similarly, the specific entropy must be conserved along a fluid streamline, implying that it must be advected along with the flow. As a result, we note that

$$\frac{\partial s}{\partial t} + \mathbf{v} \cdot \nabla s = 0. \tag{4}$$

Eqs. (3) and (4) possess definitive geometric meanings: they represent the Lie dragging of a scalar density and a scalar respectively [25]. In some cases, the entropy density $\sigma = \rho s$ is employed, allowing us to treat it on the same footing as the density, and giving rise to a conservation law akin to (3). We see that this is consistent with the physical intuition that ρ and s are scalars, and that the former also represents a density. Alternatively, when seen in the light of two dimensions, the equations for ρ and s can easily be interpreted as the Lie-dragging of a 2-form and a 0-form (in 2D) respectively [26]. We need to now determine the conservation law for ℓ. As it possesses the units of angular momentum density, we may postulate that it possesses a similar conservation law to that of density, i.e. akin to mass conservation, we introduce an angular momentum conservation law. Hence, we note that it obeys

$$\frac{\partial \ell}{\partial t} + \nabla \cdot (\ell \mathbf{v}) = 0. \tag{5}$$

We note that ℓ serves either as an independent dynamical variable, or as a function of ρ and s and in either case is independent of the velocity $\mathbf{v}$. Hence, it is very different from the usual angular momentum density $\mathbf{r} \times (\rho \mathbf{v})$ and can thus be described as an *intrinsic* angular momentum.

Each of these variables, along with $\mathbf{v}$, are functions of $\mathbf{r}$ and t. We can now construct the appropriate hydrodynamic action by writing down the appropriate Lagrangian density:

$$\mathcal{L}_H = \frac{1}{2} \rho v^2 - \rho U(\rho, s). \tag{6}$$

The first term represents the kinetic energy density, and the second represents the internal energy of the fluid, expressed in terms of the thermodynamic variables ρ and s. The Lagrangian density (6) has been shown to give rise to the ideal fluid equation of motion [22]. To this term, we shall now add a new one, and the final Lagrangian density is given by

$$\mathcal{L} = \mathcal{L}_H - \frac{1}{2} \epsilon_{ki} v_i \partial_k \ell, \tag{7}$$

where ϵ_{ki} is the 2D Levi-Civita tensor. The inclusion of the additional term appears purely phenomenological, and we label this *Issue II*, and provide a discussion as to why it is justified on physical grounds. Note that $\int \epsilon_{ki} v_i \partial_k \ell \, d^2 r = -\int \epsilon_{ki} \ell \partial_k v_i \, d^2 r$ since we have integrated by parts, and neglected the boundary term. We shall repeat this procedure in this Letter consistently henceforth.

In order to obtain the Euler–Lagrange equations of motion, we need to vary the action $S = \int \mathcal{L} \, d^2 r \, dt$ but the variation must constitute a *constrained* variation; in other words, the variation with